24. *Inversive Semigroups. II*

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This paper is the continuation of the previous paper Yamada [1]. Any terminology without definition should be referred to [1]. In this paper, we shall present necessary and sufficient conditions for an inversive semigroup to be isomorphic to some special subdirect product of a group and a band.

The proofs of any theorems and corollaries are omitted and will be given in detail elsewhere.[1)]

§1. **Group-semilattices.** Let G be a group, and Γ a semilattice. Let $\{G_\gamma : \gamma \in \Gamma\}$ be a collection of subgroups G_γ of G such that

(1) $\bigcup\{G_\gamma : \gamma \in \Gamma\} = G$

and (2) if $\alpha \leqq \beta$ (i.e., $\alpha\beta = \beta\alpha = \alpha$) then $G_\alpha \supset G_\beta$.

Let $S = \sum_{\gamma \in \Gamma} G_\gamma$, where $\sum$ denotes the class sum (i.e., the disjoint sum) of sets. If $x \in G$ is an element of G_γ, then we denote x by (x, γ) when we regard x as an element of G_γ in S. Now, S becomes a semigroup under the multiplication $\circ$ defined by the following

(P) $(x, \alpha) \circ (y, \beta) = (xy, \alpha\beta)$.

That is, S is a compound semigroup of $\{G_\gamma : \gamma \in \Gamma\}$ by Γ,[2)] and accordingly a (C)-inversive semigroup. We shall call such an S a *group-semilattice* of G, and denote by $\{G_\gamma | \Gamma, G\}$. Moreover, in this case we shall call G the *basic group* of S. Now, let I be a band whose structure decomposition is $I \sim \sum \{I_\gamma : \gamma \in \Gamma\}$.[3)] Then, we can consider the spined product of S and I with respect to Γ, because S and I have the same structure semilattice Γ.

As a connection between subdirect products of G and I and the spined product of S and I, we have

Theorem 1. *The spined product of group-semilattice of G and a band I is isomorphic to an inversive subdirect product of G and I.*[4)] *Conversely, any inversive subdirect product of a group G and a band I is isomorphic to the spined product of a group-semilattice of G and I.*

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1) This is an abstract of the paper which will appear elsewhere.

2) For compound semigroups, see M. Yamada, Compositions of semigroups, Kōdai Math. Sem. Rep., **8**, 107–111 (1956).

3) For the definition of the structure decomposition of a band, see N. Kimura, Note on idempotent semigroups. I, Proc. Japan Acad., **33**, 642–645 (1954).

4) Let D be a subdirect product of G and I. Then, D is clearly a semigroup. If D is an inversive semigroup, then D is called an *inversive subdirect product* of G and I.

Corollary. *An inversive semigroup is isomorphic to a subdirect product of a group G and a band I if and only if it is isomorphic to the spined product of a group-semilattice of G and I.*

Remark. Let G be a group, and I a band whose structure decomposition is $I\sim\sum\{I_\gamma:\gamma\in\Gamma\}$. Then, the direct product of G and I is isomorphic to the spined product of $\{G_\gamma|\Gamma, G\}$ and I, where $G_\gamma=G$ for all γ, and vice versa.

By Theorem 1 and its remark, the connection between direct products, subdirect products and spined products is somewhat clarified. These results will be used in the next paragraph.

§2. Necessary and sufficient conditions for an inversive semigroup to be isomorphic to the spined product of a group-semilattice and a band.

The following is a well-known result: If C_1, C_2 are congruences on an algebraic system A such that

(S. 1) $C_1\cap C_2=0$,

then A is isomorphic to a subdirect product of A/C_1 and A/C_2. Further, if C_1, C_2 are permutable congruences and if they satisfy (S. 1) and

(S. 2) $C_1\cup C_2=1$,

then A is isomorphic to the direct product of A/C_1 and A/C_2.

Using Theorem 1, its corollary and the result in above, we have

Theorem 2. *An inversive semigroup S is isomorphic to the spined product of a group-semilattice and a band if and only if the following relations R_1, R_2 are congruences on S:*

(1) *aR_1b if and only if ab^{-1} and ba^{-1} are idempotents.*

(2) *aR_2b if and only if $aa^{-1}=bb^{-1}$.*

Further, if S is isomorphic to the spined product of a group-semilattice L and a band B, then the basic group of L and the band B are isomorphic to S/R_1 and S/R_2 respectively. Accordingly, in this case S is also isomorphic to a subdirect product of S/R_1 and S/R_2.

Remark. In Theorem 2, let I be the subband consisting of all idempotents of S. Then, S/R_2 is also isomorphic to I.

Corollary. *An inversive semigroup S is isomorphic to the direct product of a group and a band if and only if R_1, R_2 are permutable congruences on S and satisfy the condition*

(S. 2) $R_1\cup R_2=1$.

Moreover, Theorem 2 is paraphrased as follows:

Theorem 3. *Let S be an inversive semigroup, and I the subband consisting of all idempotents of S. Then, S is isomorphic to the spined product of a group-semilattice and a band if and only if it satisfies the following* (C. 1) *and* (C. 2):

(C. 1) *S is strictly inversive.*

(C. 2) *For any* $e \in I$, $ab \in I$ *if and only if* $aeb \in I$.

Corollary. *Let* S *be an inversive semigroup, and* I *the subband consisting of all idempotents of* S. *Then,* S *is isomorphic to the direct product of a group and a band if and only if it satisfies the conditions* (C. 1), (C. 2), *and the following* (C. 3):

(C. 3) *For any* $a, b \in S$, *there exist* x, y *such that* $aa^{-1} = xx^{-1}$, $bb^{-1} = yy^{-1}$ *and* $xb^{-1}, ya^{-1}, bx^{-1}, ay^{-1} \in I$.

Reference

[1] M. Yamada: Inversive semigroups. I, Proc. Japan Acad., **39**, 100-103 (1963).