21. *Normality and Perfect Mappings*

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We assume that the spaces considered here are always completely regular T_1-spaces. A mapping φ from X onto Y is said to *be perfect* if φ is a closed continuous mapping and every $\varphi^{-1}(y)$, $y \in Y$, is compact, i.e., φ is a compact mapping. Let E be any dense subspace of a given space X. It is easy to see that the normality of $X \times \beta E$ implies the normality of $X \times BE$ where BE is any compactification of E and βE is the Stone-Čech compactification of E. But the following problem is open [1, § 4].

(*) *Does the normality of* $X \times BE$ *implies the normality of* $X \times \beta E$?

This problem is closely related to the following open problem [1, problem 4]:

(**)[1] *Let* φ *be a perfect mapping from* X *onto* Y *such that the image of any proper closed subset of* X *is a proper closed subset of* Y. *Is it true that* X *is normal whenever* Y *is normal?*

In § 1, we shall investigate some special class of spaces, and, in § 2, we shall give the negative answers to the problems (*) and (**).

In the sequel, ω_α denotes the smallest ordinal of cardinal $\aleph_\alpha$ and we mean by $W(\omega_\alpha)$ the set of all cardinals less than ω_α; then $W(\omega_\alpha)$ $(\alpha \neq 0)$, endowed the interval topology, is a countably compact normal space and there are no subsets of cardinal $< \aleph_\alpha$ which are cofinal [6, 9K].

1. Closedness of projections. We mean by φ(or φ_X): $X \times Y \to X$ the projection $\varphi(x, y) = x$ from $X \times Y$ onto X. Let $\mathfrak{N}$ be the class consisting of all X such that $\varphi: X \times Y \to X$ is always closed for any countably compact space Y.

1.1. Lemma. *If* X *has the property such that for any point* p *and any subset* E *of* X, *there is a sequence in* E *converging to* p *whenever* p *is an accumulation point of* E, *then* X *belongs to* $\mathfrak{N}$.

Proof. Let Y be a countably compact space and F a closed subest of $X \times Y$ such that the image E of F under $\varphi: X \times Y \to X$ is not closed. There is a point p in $\overline{E} - E$. By the assumption, there is a sequence (x_n) in E converging to p. Let (x_n, y_n) be a point of F for every n. Since Y is countably compact, there is an accumula-

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1) This problem is raised by Nagami [7] in connection with Ponomarev's theorem [8].

tion point y of $\{y_n\}$. It is easy to see that (p, y) is an accumulation point of $\{(x_n, y_n)\}$ and hence $(p, y) \in F$ because F is closed. This shows that $p \in F$. This is a contradiction.

From 1.1, if either X satisfiies the first axiom of countability or X is the one-point compactification of a discrete space, then X belongs to $\mathfrak{N}$. Moreover it is easily seen that if $X \in \mathfrak{N}$, then any subspace X belongs to $\mathfrak{N}$ and the image of X under a closed mapping belongs to $\mathfrak{N}$.

1.2. **Lemma.** *If $X \in \mathfrak{N}$ and E is a countably compact subset of X, then E is closed.*

Proof. Suppose that E is not closed. Let us put $\Delta = \{(e, e); e \in E\}$ and $\varphi: X \times E \to X$. The subset Δ is closed in $X \times E$, but $\varphi(\Delta) = E$ is not closed in X. This is a contradiction.

1.3. **Theorem.** *Let X be a locally compact space: then $X \in \mathfrak{N}$ if and only if any countably compact subset of X is closed.*

Proof. Necessity follows from Lemma 1.2. To prove the sufficiency, suppose that $\varphi: X \times Y \to X$ is not closed for some countably compact space Y. Let F be a closed subset of $X \times Y$ such that $E = \varphi(F)$ is not closed, that is, there is a point p in $\bar{E} - E$. Since X is locally compact, there is a compact neighborhood U of p and $F_U = F \frown (U \times Y)$ is closed and hence a countably compact subset of $U \times Y$ since $U \times Y$ is countably compact. Thus $\varphi(F_U)$ is countably compact and hence it is closed in X by the assumption. Therefore we have $p \in \varphi(F_U) \subset \varphi(F) = E$. This is a contradiction.

1.4. **Theorem.** *Let X be a paracompact locally compact space: then $X \in \mathfrak{N}$ if and only if $X \times Y$ is normal for any countably compact normal space Y.*

Proof. Sufficiency. Let Y be a countably compact normal space. Suppose that there is a closed subset F in $X \times Y$ whose image E under $\varphi: X \times Y \to X$ is not closed, and hence there is a point p in $\bar{E} - E$. Since E is locally compact, there is a compact neighborhood U of p and $U \times Y$ is countably compact. A mapping $\varphi_U: U \times Y \to U$ is a Z-mapping, that is, any zero-set of $U \times Y$ is mapped onto a closed subset of U by φ_U[2]. Since $U \times Y$ is normal, it is easy to see that φ_U is closed. This means that $p \in \varphi_U(F \frown (U \times Y)) \subset \varphi(F) = E$. This is a contradiction.

Necessity follows from the following theorems: let f be a closed mapping from Z onto a paracompact space X, then the followings are equivalent: 1) Z is normal, 2) $f^{-1}(x)$ is normal and normally embedded in Z for each $x \in X$ [3] and 3) for each $x \in X$, any two disjoint closed subset of $f^{-1}(x)$ can be separated by open sets of Z [4].

2. **Answers to problems.** **2.1. Problem** (*). Let $X = X(\omega_\alpha)(\alpha \neq 0)$, E the discrete subset of X consisting of all non-limit ordinals and

let $M=E\smile\{p\}$ be the one point compactification of E. By 1.1 and 1.4, $X\times M$ is normal. On the other hand $X\times\beta E$ is not normal. For if $X\times\beta E$ is normal, then $X\times W(\omega_\alpha+1)=T$ is normal because T is the image of $X\times\beta E$ under the closed mapping ψ constructed by the following way; let φ be a Stone extension mapping from βE onto $W(\omega_\alpha+1)$ of the identity mapping on E, and ψ be a mapping: $\psi(x, y)=(x, \varphi(y))$, $x\in X$, $y\in\beta E$; then φ is compact and closed, and hence ψ becomes to be closed [5]. On the other hand, we proved that the product $Y\times Z$ of a non-compact countably compact normal space Y with its any compactification Z is not normal [9]. Thus $X\times\beta E$ is not normal which is a negative answer to problem (*).

2.2. **Problem** (**). We shall use E and X in 2.2. Let f be a Stone extension mapping from βE onto M of the identity mapping on E. Then a mapping $\varphi(x, y)=(x, f(y))$, from $X\times\beta E$ onto $X\times M$ is a perfect mapping [5]. Let F be a proper closed subset of $X\times\beta E$. Since $U=X\times\beta E-F$ is open and $X\times E$ is dense in $X\times\beta E$, U contains a point $a=(x, e)$, where $x\in X$ and $e\in E\subset\beta E$. Then $\varphi^{-1}(x, e)$ $(x\in X, e\in E\subset M)$ consists of only one point a. Thus $\varphi(F)\neq X\times M$, i.e., φ is a proper mapping. This is a negative answer to problem (**).

2.3. *The Stone-Čech compactification of a discrete space does not belong to* $\mathfrak{N}$.

Proof. Any discrete space is homeomorphic to the set of all non-limit ordinal of a space $W(\omega_\alpha)$ for a suitable $\aleph_\alpha$. We have 2.3 by the method used as in the proof of 2.1.

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