19. The ε-Entropy of Some Classes of Harmonic Functions

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1. Let K be a bounded continuum in q-dimensional Euclidian space and G be a bounded open set containing K. For complex-valued function $u(x)$ in G, we define $\|u(x)\|=\sup_{x\in K}|u(x)|$. We consider classes $H_G(C)$ of functions $u(x)$ which are harmonic in G and bounded in G by the constant C. When we introduce the metric $\|\cdot\|$ in $H_G(C)$, we shall denote it by $H_G^K(C)$.

The purpose of the present paper is to compute "ε-entropy" and "ε-capacity" of $H_G^K(C)$ for some K and G. The exact formulae for them are given in 3. Using these results, we can compute the "functional dimension" of the vector space of harmonic function in 4.

The problem of computing ε-entropy of the space of solutions of partial differential equations was posed by Prof. H. Yoshizawa.

2. Following [3], we shall list definitions which are necessary to state our results. Let R be a metric space and A a set in R.

DEFINITION 1. A set B in R is called an ε-net for the set A if every points of A is at a distance not exceeding ε from some point of B.

DEFINITION 2. A set B in R is called ε-separated if the distance of any distinct points of B are greater than ε.

Now we assume the set A is totally bounded.

DEFINITION 3. $N(\varepsilon, A)$ is the minimal number of points in all possible ε-net for A. $H(\varepsilon, A)=\log N(\varepsilon, A)$ is called ε-entropy of the set A. ($\log N$ will always denote the logarithm of the number N in the base 2.)

DEFINITION 4. $M(\varepsilon, A)$ is the maximal number of points in all possible ε-separated subsets of the set A. $C(\varepsilon, A)=\log M(\varepsilon, A)$ is called the ε-capacity of A.

We shall state a simple theorem which will be used later [3].

THEOREM 1. $M(2\varepsilon, A)\leqq N(\varepsilon, A)$

3. Our result is as follows.

THEOREM. Let $K_r=\{x; \sum_{i=1}^q x_i^2\leq r^2\}$ and $G_R=\{x; \sum_{i=1}^q x_i^2<R^2\}$ in q-dimensional space. Then

$$H(\varepsilon,\ H_{G_R}^{K_r}(C))=\{4/q!\,(\log R/r)^{q-1}\}\,(\log 1/\varepsilon)^q+O((\log 1/\varepsilon)^{q-1}\log\log 1/\varepsilon),$$
$$C(2\varepsilon, H_{G_R}^{K_r}(C))=\{4/q!\,(\log R/r)^{q-1}\}\,(\log 1/\varepsilon)^q+O((\log 1/\varepsilon)^{q-1}\log\log 1/\varepsilon).$$

(For notations, see 1 and 2.)

REMARK. From Theorem 1 it is sufficient to estimate $H(\varepsilon, A)$ from above (formula in 8) $C(2\varepsilon, A)$ from below (formula in 10).

4. Let G be an arbitrary domain in q-dimensional space. H_G is the totality of harmonic function in G. We introduce in H_G compact uniform topology and consider it as linear topological space. The functional dimension of a linear topological space Φ is defined as follows: ([2])

$$\text{df}\,\Phi=\sup_U \inf_V \overline{\lim_{\varepsilon\to 0}} \log\log N(\varepsilon, V/U)/\log\log 1/\varepsilon$$

where $N(\varepsilon, V/U)=\inf\{N;\ V\subset\bigcup_{k=1}^N(\varphi_k+U),\ \varphi_k\in\Phi\}$ and inf and sup are taken for all neighbourhoods of 0 in Φ.

Then we obtain $\quad \text{df}\,H_G=q.$

In order to compute $\text{df}\,H_G$ we use our results and the following properties for $N(\varepsilon, H_G^K(C))$ which can be proved easily:

$$N(\varepsilon, H_{G_1}^{K_1}(C))\leq N(\varepsilon, H_{G_2}^{K_2}(C)) \text{ if } K_1\subset K_2 \text{ and } G_1\supset G_2$$
$$N(\varepsilon, H_G^{K_1\cup K_2}(C))\leq N(\varepsilon, H_G^{K_1}(C))N(\varepsilon, H_G^{K_2}(C)).$$

5. We shall prove our THEOREM in 5–10. First we shall consider hyperspherical harmonics for the later use. Function $u(x)=u(\rho, s)$ of the class $A=H_{G_R}^{K_r}(C)$ can be expanded in hyperspherical harmonics in K_r ([1]).

$$(1)\qquad \begin{cases} u(\rho, s)=\sum_{l=0}^{\infty}(2l+p)\,(\rho/r)^l u_l(s) \\ u_l(s)=\{\Gamma(p/2)/4\pi^{p/2+1}\}r^{-p-1}\displaystyle\int_{S(r)} u(\rho, s')V_l^{(p)}(\cos\gamma)\,ds' \end{cases}$$

where $q=p+2$, $S(r)$ is the sphere of radius r, $\gamma=\angle sOs'$ and $(1-2ax+a^2)^{-p/2}=\sum_{l=0}^{\infty}a^l V_l^{(p)}(x)$.

We list here some properties of the above expansion for later use

(A) We have $|V_l^{(p)}(\cos\gamma)|\leq c_l$, where $c_l=V_l^{(p)}(1)=(l, p)/(1, p)$, $(\lambda, k)=\Gamma(\lambda+k)/\Gamma(\lambda)=\lambda(\lambda+1)\cdots(\lambda+k-1)$.

(B) Hyperspherical functions of order l form a d_l-dimensional vector space H_l, where $d_l=\{(l+1, p-1)/(1, p)\}\cdot(2l+p)$.

(C) We have $\int V_l^{(p)}(\cos\angle NOs)^2\,ds=\{4\pi^{p/2+1}/\Gamma(p/2)\}\cdot c_l/2l+p$.

LEMMA. If $y_l(s)\in H_l$ and $\int|y_l(s)|^2\,ds=1$, then

$$(2)\qquad |y_l(s)|\leq C_1\{(2l+p)c_l\}^{\frac{1}{2}}$$

and C_1 does not depend on l (C_i will always mean constants which depend only on p, r, R, C).

PROOF. Put $u(\rho, s)=\rho^l y_l(s)$ in (1). If we use Schwartz' inequality and (C), we get (2).

6. We define a norm for bounded functions on unit sphere by $||u(s)||'=\sup_{s\in S(1)}|u(s)|$. Then we get two inequalities for expansion (1).

$$(3)\qquad ||u_l(s)||'\leq C_2c_l'||u(\rho, s)||, \text{ where } c_l'=\{c_l/2l+p\}^{\frac{1}{2}}$$

$$(4)\qquad ||u(\rho, s)||\leq\sum_{l=0}^{\infty}(2l+p)\,||u_l(s)||'.$$

Because $u(\rho, s)\in A$ is harmonic in G_R, it has an expansion of the form (1) in $K_{R'}(R'<R)$, where r is substituted by R'. By equating ρ^{l}'s coefficients in this expansion and in the original one, we get

$$(5)\qquad u_l(s)=\{\Gamma(p/2)/4\pi^{p/2}+1\}\cdot R'^{-p-1}(r/R')^l\int_{S(R')} u(\rho, s')V_l^{(p)}(\cos\gamma)\,ds'.$$

We obtain from (5) and $|u(\rho, s)|\le C$ in G_R, $\|u_l(s)\|'\le C_3c_l'(r/R')^l$. Because $R'<R$ is arbitrary, we get finally

$$(6)\qquad \|u_l(s)\|'\le C_3\,c_l'\,e^{-hl},\ \text{where } e^h=R/r.$$

7. We define n as the smallest number that satisfies $\sum_{l=n}^{\infty}(2l+p)\times C_3\,c_l'\,e^{-hl}\le\varepsilon/2$. Because left side of the above inequality is smaller than $C_4\,n^N e^{-hn}$ for some N, we get the following estimation of n:

$$(7)\qquad n=\log 1/\varepsilon/h\log e+O(\log\log 1/\varepsilon).$$

For such n, if we define $\hat{u}(\rho, s)$ by $\hat{u}(\rho, s)=\sum_{l=0}^{n-1}(2l+p)(\rho/r)^l u_l(s)$ then $\hat{A}=\{\hat{u};\,u\in A\}$ is an $\varepsilon/2$-net for A. We define A_l by $\{u_l(s);\,u(\rho, s)\in A\}$. If we put $\varepsilon'=\varepsilon/2/n(n+p-1)$ and if we construct ε'-net B_l for A_l in H_l (in metric $\|\cdot\|'$), then $\{\sum_{l=0}^{n-1}(2l+p)(\rho/r)^l u_l(s);\,u_l\in B_l\}$ will be $\varepsilon/2$-net for $\hat{A}$, so ε-net for A.

If number of elements B_l is N_l, $N(\varepsilon, A)\le\prod_{l=0}^{n-1}N_l$.

8. We construct B_l and evaluate N_l. Let $\{y_k^l(s),\ 1\le k\le d_l\}$ be complete orthonormal system in H_l, and we shall expand $u_l(s)\in A_l$ in $\{y_k^l(s)\}$:

$$u_l(s)=\sum_{k=1}^{d_l}b_k^l y_k(s),\ \text{where } b_k^l=\int_S u_l(s)\overline{y_k^l(s)}\,ds.$$

From (6), we obtain for $u_l(s)\in A_l$

$$(8)\qquad |b_k^l|\le C_4\,c_l'\,e^{-hl}.$$

If we consider the class of elements of H_l, whose b_k^l can be written as $b_k^l=m_k^l\delta+m_k'^l\delta\sqrt{-1}$ (where m_k^l, $m_k'^l$ are integers, and $\delta=(2\varepsilon'/\sqrt{2})/d_lC_1\{(2l+p)\,c_l\}^{\frac{1}{2}}$), then from the lemma, it is an ε'-net for A_l.

From (8), it is sufficient to choose $|m_k^l|\le C_4\,c_l'\,e^{-hl}/\delta$. So

$$N_l\le\{2[C_4\,c_l'\,e^{-hl}/\delta]\}^{2d_l}.$$

$$H(\varepsilon, A)\le\sum_{l=0}^{n-1}\log N_l=\sum_{l=0}^{n-1}2d_l\log(C_5\,n(n+p-1)\,d_l\,c_l\,e^{-hl}/\varepsilon)$$

$$=\frac{4}{(p+2)!}\cdot(\log 1/\varepsilon)^{p+2}/(h\log e)^{p+1}\,O\{(\log 1/\varepsilon)^{p+1}\log\log 1/\varepsilon\}.$$

9. We now derive lower estimate for $C(2\varepsilon, A)$. For this purpose we use two facts:

α) A constant C_6 can be taken such that

$$(9)\qquad |b_k^l|\le C_6\{1/d_l\,(2l+p)^{\frac{3}{2}}c_l^{\frac{1}{2}}\}\Delta\,e^{-(h+\Delta)l},\ \Delta>0$$

implies $u(x)=u(\rho, s)\in A$, where

$$(10)\qquad \begin{cases} u(\rho, s)=\sum_{l=0}^{\infty}(2l+p)(\rho/r)^l u_l(s),\\ u_l(s)=\sum_{k=1}^{d_l}b_k^l\,y_k^l(s).\end{cases}$$

Proof. Under the assumption on b_k^l, from lemma we obtain $\|u_l(s)\|'\le C_6\cdot C_1\cdot\Delta\,e^{-(h+\Delta)l}/2l+p$. We have, in G,

$|u(\rho, s)| \leq \sum_{l=0}^{\infty}(2l+p)(R/r)^l \|u_l(s)\|' \leq C_6 \cdot C_1 \sum_{l=0}^{\infty} \Delta e^{-\Delta l} = C_6 \cdot C_1 \cdot \Delta/1-e^{-\Delta}$.
This can be made $\leq C$, where C is independent of Δ.

β) In expansion (10), we have

$$|b_k^l| \leq C_7 c_l' \|u\|. \tag{11}$$

This is a consequence of $|b_k^l| \leq \|u_l(s)\|'$ and (3).

10. Now put $\Delta = h/\log 1/\varepsilon$ and fix n (how to take n will be shown later). The set of $u(\rho, s) = \sum_{l=0}^{n}(2l+p)(\rho/r)^l u_l(s)$ is a 2ε-separated subset of A, if $u_l(s) = \sum_{k=1}^{d_l}(s_k^l + \sqrt{-1}\, s'^l_k) 2\varepsilon C c_l' y_k^l(s)$ where s_k^l, s'^l_k are integers which satisfy

$$|s_k^l| \leq (1/\sqrt{2})\, C_6 \{1/d_l (2l+p)^{\frac{3}{2}} c_l^{\frac{1}{2}}\} \Delta\, e^{-(h+\Delta)l} / 2\varepsilon \cdot C_6 \cdot c_l'. \tag{12}$$

Now n is defined as the largest of natural numbers l that make right hand side of (12) not smaller than 1. Then n can be estimated as follows: $n = \log 1/\varepsilon / h \log e + O(\log\log 1/\varepsilon)$.

If we put $M_l^k = 2[(1/\sqrt{2})\, C_6\{1/d_l(2l+p)^{\frac{3}{2}} c_l^{\frac{1}{2}}\} \Delta e^{-(h+\Delta)l}/2\varepsilon \cdot C_7 \cdot c_l'] + 1$, we get

$$M(2\varepsilon, A) \geq \prod_{l=0}^{n} \prod_{k=1}^{d_l} M_l^{k^2}.$$

Hence

$$C(2\varepsilon, A) \geq \sum_{l=0}^{n} 2d_l \log (C_8\{1/d_l\,(2l+p) c_l \cdot \varepsilon\} \Delta\, e^{-(h+\Delta)l})$$
$$= \{4/(p+2)!\} (\log 1/\varepsilon)^{p+2} / (h \log e)^{p+1} + O\{(\log 1/\varepsilon)^{p+1} \log\log 1/\varepsilon\}.$$

References

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