18. *On Rings of Analytic Functions on Riemann Surfaces*

By Mitsuru NAKAI
Mathematical Institute, Nagoya University

(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1963)

Let R_j be an open Riemann surface and $A(R_j)$ be the ring of all one-valued regular analytic functions on R_j $(j=1, 2)$ and σ be a ring isomorphism of $A(R_1)$ onto $A(R_2)$. Since the imaginary unit i is the primitive fourth root of 1, either $i^\sigma=i$ or $-i$. In the former (resp. latter) case, σ is called a *direct* (resp. *indirect*) ring isomorphism. Suppose that there exists a one-to-one transformation S of R_1 onto R_2. If S is directly conformal, then S induces a direct ring isomorphism σ defined by the relation

$$f(p)=f^\sigma(S(p)) \quad (f\in A(R_1),\ p\in R_1).$$

If S is indirectly conformal, then S induces an indirect ring isomorphism σ defined by the relation

$$\overline{f(p)}=f^\sigma(S(p)) \quad (f\in A(R_1),\ p\in R_1).$$

In either case, we say that *σ is induced by S*. The aim of this note is to prove the converse of the above fact.

Theorem. *Any direct (resp. indirect) ring isomorphism of $A(R_1)$ onto $A(R_2)$ is induced by a unique one-to-one direct (resp. indirect) conformal transformation of R_1 onto R_2.*

This fact is first proved by Bers under the assumption that R_1 and R_2 are open plane domains.[1] For arbitrary open Riemann surfaces R_1 and R_2, Rudin proved the above fact under the assumption that the given isomorphism preserves complex constants unchanged.[2] Hence our Theorem, in which no *a priori* assumption on complex constants is made, is a proper generalization of Bers' result and also contains Rudin's result.[3] We divide the proof of our Theorem into several lemmas. Some of them are well known but we include their proofs for the sake of completeness.

1. Ring isomorphism on complex numbers. Let σ be the given ring isomorphism of $A(R_1)$ onto $A(R_2)$ and τ be the inverse of σ. The map τ is also a ring isomorphism of $A(R_2)$ onto $A(R_1)$. We denote by C the complex number field and by C_r the complex rational number field, where a complex number, both of whose real and ima-

1) Bull. Amer. Math. Soc., **54**, 311–315 (1948).

2) Bull. Amer. Math. Soc., **61**, 543 (1955).

3) This problem is suggested by Prof. Bers. If $R_1 \notin O_{AB}$, then our Theorem is easily reduced to Rudin's result. See Proposition 3 in Royden's paper: Seminars on analytic functions, Inst. for advanced study, Princeton, **2**, 273–285 (1958).

ginary part is a real rational number, is called a complex rational number. Clearly C_r and C are subrings of $A(R_j)$ $(j=1,2)$.

Lemma 1. *If σ is direct (resp. indirect), then for any α in C_r*

$$\alpha^\sigma=\alpha^\tau=\alpha \ (resp.\ \bar{\alpha}).$$

Proof. Since C_r is generated by 1 and i, our assertion is clear if we notice that $1^\sigma=1^\tau=1$ and $i^\sigma=i^\tau=i$ (resp. $-i$).

Sublemma 2.1. *The following two assertions are equivalent:*

(a) *a function f in $A(R_j)$ belongs to C;*

(b) *for any α in C_r, there exists a function f_α in $A(R_j)$ such that*

$$f-\alpha=f_\alpha^2.$$

Proof. The implication (a)→(b) is trivial. To show the implication (b)→(a), contrary to the assertion, assume that f is not a constant. Then we can find a subdomain U in R_j in which f is univalent. Since $f(U)$ is open in C and C_r is dense in C, there exists a point p in U such that $\alpha=f(p)$ belongs to C_r. By the assumption (b), there exists a function f_α in $A(R_j)$ such that $f=\alpha+f_\alpha^2$. Since $f_\alpha(p)=0$, f is not univalent in U near p. This is a contradiction.

Lemma 2. $C^\sigma=C$ *and* $C^\tau=C$.

Proof. In virtue of Lemma 1, the property (b) in Sublemma 2.1 is preserved by σ and τ. Thus by Sublemma 2.1, σ and τ carry constants to constants. Hence $C^\sigma=C^\tau=C$.

Remark. we shall later see that σ and τ are trivial ring automorphisms of C onto itself, i.e. $\alpha^\sigma=\alpha^\tau=\alpha$ (resp. $\bar{\alpha}$). But at this stage, we cannot yet conclude this. In fact, there really exist infinitely many non-trivial ring automorphisms of C onto itself.

2. Principal ideals. Let f be in $A(R_j)$. We denote by (f) the set

$$(gf;\ g\in A(R_j)),$$

if it does not contain the constant 1. The set (f) is an ideal of $A(R_j)$ and called a *principal ideal* (abbreviated as *p.i.*) of $A(R_j)$. We say that a *p.i.* (f) is a *maximal principal ideal* (abbreviated as *m.p.i.*) if for any *p.i.* (g) such that $(f)\subset(g)$, we get $(f)=(g)$.

For each point p in R_j, we denote by J_p the set

$$(g;\ g\in A(R_j),\ g(p)=0),$$

which is also an ideal of $A(R_j)$.

Sublemma 3.1. (a) *For any m.p.i. (f), there exists a point p in R_j such that $(f)=J_p$.*

(b) *For any point p in R_j, there exists an m.p.i. (f) such that $J_p=(f)$.*

Proof. Ad. (a): Let (f) be an *m.p.i.* The function f vanishes at some point of R_j. If this is not the case, then by $1/f\in A(R_j)$, (f) would contain the constant $1=(1/f)f$, which is a contradiction. Let

p be one of the zero points of f. By Florack's theorem,[4] we can find a function h in $A(R_j)$ such that $g=f/h$ belongs to $A(R_j)$ and has only a simple zero at p. Then clearly (g) is a *p.i.* and $(f)\subset(g)$. Hence by the maximality of (f), $(f)=(g)$ or $g\in(f)$. Thus $g=kf$ for some k in $A(R_j)$ and so f has only a simple zero at p. From this, it follows that $(f)=J_p$.

Ad. (b): By Florack's theorem,[4] there exists a function f in $A(R_j)$ which has only a simple zero at p. Clearly (f) is an *m.p.i.* and $(f)=J_p$.

Sublemma 3.2. *There exists a one-to-one mapping S (resp. T) of R_1 (resp. R_2) onto R_2 (resp. R_1) such that $S^{-1}=T$ and*

$$J_p^\sigma=J_{S(p)} \qquad (\text{resp. } J_q^\tau=J_{T(q)}).$$

Proof. For each point p in R_1, there exists an *m.p.i.* (f) such that $J_p=(f)$ (Sublemma 3.1). Clearly $J_p^\sigma=(f^\sigma)$ is an *m.p.i.* in $A(R_2)$ and so there exists a point q in R_2 such that $(f^\sigma)=J_q$ (Sublemma 3.1). We define the mapping S by $q=S(p)$, i.e.

$$J_p^\sigma=J_{S(p)} \quad (p\in R_1).$$

It is easy to see that S is a one-to-one mapping of R_1 onto R_2. Similarly, we can define the desired mapping T. Since

$$J_{TS(p)}=(J_{S(p)})^\tau=(J_p^\sigma)^\tau=J_p,$$

it holds that $TS(p)=p$ on R_1. Similarly, $ST(q)=q$ on R_2. Thus $S^{-1}=T$.

Lemma 3. *There exists a one-to-one mapping S (resp. T) of R_1 (resp. R_2) onto R_2 (resp. R_1) such that $S^{-1}=T$ and*

$$(f(p))^\sigma=f^\sigma(S(p)) \qquad (f\in A(R_1),\ p\in R_1)$$

and

$$(g(q))^\tau=g^\tau(T(q)) \qquad (g\in A(R_2),\ q\in R_2).$$

Proof. Let S and T be as in Sublemma 3.2. Since $f-f(p)$ belongs to J_p, the function $f^\sigma-(f(p))^\sigma$ which is equal to $(f-f(p))^\sigma$ belongs to $J_p^\sigma=J_{S(p)}$. Thus $f^\sigma(S(p))=(f(p))^\sigma$. Similarly, we get the identity for τ and T.

3. Continuity properties. A mapping of a topological space into another is called a *compact mapping* if the closure of the image of any compact set under this mapping is again compact.

Lemma 4. *The mapping S (resp. T) in Lemma 3 is a compact mapping.*

Proof. Let K be an arbitrary compact set in R_1. We have to show that the set $\overline{S(K)}$ is a compact set in R_2. Contrary to the assertion, assume that $\overline{S(K)}$ is not compact. Then there exists an infinite sequence (q_n) of distinct points in $S(K)$ which does not accumulate in R_2. Set $p_n=T(q_n)$. Then (p_n) is a sequence of points

4) Schr. Math. Inst. Univ. Münster, no. **1** (1948).

in the compact set K. Hence by choosing a suitable subsequence, we may assume that (p_n) itself converges to a point p_0 in K. By Florack's theorem,[4)] we can find a function g in $A(R_2)$ such that

$$g(q_n)=n \quad (n=1,2,3,\cdots).$$

Put $f=g^\tau$. Then by Lemma 3,

$$(f(p_n))^\sigma=f^\sigma(S(p_n))=g(q_n)=n \quad (n=1,2,3,\cdots).$$

Hence by Lemma 1,

$$f(p_n)=((f(p_n))^\sigma)^\tau=n^\tau=n \quad (n=1,2,3,\cdots).$$

Thus we arrive at the following contradiction:

$$f(p_0)=\lim_{n\to\infty}f(p_n)=\infty.$$

Hence $\overline{S(K)}$ is compact and so S is a compact mapping. Similarly, we can show that T is a compact mapping.

Sublemma 5.1. *The mappings τ and σ are continuous on C with respect to the usual plane topology in C.*

Proof. Since the situation is quite parallel, we only prove the continuity of σ. For the aim, we have only to show that in C,

$$\lim_{n\to\infty}\alpha_n=0 \quad \text{implies} \quad \lim_{n\to\infty}\alpha_n^\sigma=0.$$

First we show that the sequence (α_n^σ) is bounded. For the aim, choose a function f in $A(R_1)$ and a point p in R_1 such that f has a simple zero at p. Let U be an open neighborhood of p such that $\overline{U}$ is compact and f is univalent in U. Then $f(U)$ is open in C and so contains α_n $(n\geq n_0)$ for some n_0. Then we can find a sequence (p_n) of points in U such that

$$f(p_n)=\alpha_n \quad (n\geq n_0).$$

Let $g=f^\sigma$ and $q_n=S(p_n)$ $(n\geq n_0)$. Then the sequence (q_n) is contained in the compact set $K=\overline{S(\overline{U})}$ (Lemma 4). Since

$$\sup(|g(q)|;q\in K)=\rho<\infty$$

and

$$\alpha_n^\sigma=(f(p_n))^\sigma=f^\sigma(S(p_n))=g(q_n) \quad (n\geq n_0),$$

we get

$$|\alpha_n^\sigma|\leq\rho \quad (n\geq n_0).$$

Finally we show that $\lim_{n\to\infty}\alpha_n^\sigma=0$. Assume the contrary. Then, since (α_n^σ) is bounded, there exists a subsequence of (α_n^σ) converging to a non-zero number. By renumbering, we may assume that (α_n^σ) itself converges to a non-zero number α. Let $\beta_n=1/\alpha_n$ and $\beta=1/\alpha$. Then

$$\lim_{n\to\infty}\beta_n=\infty \quad \text{and} \quad \lim_{n\to\infty}\beta_n^\sigma=\beta\neq\infty.$$

We can find a function g in $A(R_2)$ and a point q in R_2 such that the function $g-\beta$ has a simple zero at q. Let V be an open neighborhood of q such that $\overline{V}$ is compact and g is univalent in V. Since $g(V)$ is open in C and contains β, the set $g(V)$ contains β_n^σ $(n\geq n_0)$ for some n_0. So we can find a sequence (q_n) in V such that

$$g(q_n)=\beta_n^\sigma \quad (n\geq n_0).$$

Let $f=g^\tau$ and $p_n=T(q_n)$ $(n\geq n_0)$. Then the sequence (p_n) is contained in the compact set $\overline{T(\overline{V})}$ (Lemma 4). Hence, by choosing a suitable subsequence, we may assume that the sequence (p_n) converges to a point p_0 in R_1. Since

$$f(p_n)=g^\tau(p_n)=g^\tau(T(q_n))=(g(q_n))^\tau=(\beta_n^\sigma)^\tau=\beta_n,$$

we arrived at the following contradiction:

$$f(p_0)=\lim_{n\to\infty}f(p_n)=\lim_{n\to\infty}\beta_n=\infty.$$

Lemma 5. *If σ is direct (resp. indirect), then for any α in C,*

$$\alpha^\sigma=\alpha^\tau=\alpha \quad (resp.\ \bar{\alpha}).$$

Proof. For any α in C, we can find a sequence (α_n) in C_r such that $\lim_{n\to\infty}\alpha_n=\alpha$. As $\alpha_n^\sigma=\alpha_n$ (resp. $\bar{\alpha}_n$) and σ is continuous (Lemmas 1 and 4), so we get $\alpha^\sigma=\alpha$ (resp. $\bar{\alpha}$). Similarly, we get the identity for τ.

Lemma 6. *The mapping S (resp. T) in Lemma 3 is continuous.*

Proof. We show this in the case where σ is direct. The proof for indirect σ is similar. Let $p=\lim_{n\to\infty}p_n$ in R_1 and $q_n=S(p_n)$. Then (q_n) is contained in a compact set in R_2 (Lemma 4). We have to show that (q_n) is convergent. Contrary to the assertion, assume that there exist two subsequences $(q_n^{(k)})$ $(k=1,2)$ of (q_n) such that

$$\lim_{n\to\infty}q_n^{(k)}=q^{(k)} \quad (k=1,2)$$

and

$$q^{(1)}\neq q^{(2)}.$$

By Florack's theorem,[4] there exists a function g in $A(R_2)$ such that

$$g(q^{(k)})=k \quad (k=1,2).$$

Put $f=g^\tau$ and $p_n^{(k)}=T(q_n^{(k)})$ $(k=1,2;\ n=1,2,3,\cdots)$. Then

$$p=\lim_{n\to\infty}p_n^{(k)}.$$

Using Lemmas 3 and 5, we get

$$f(p_n^{(k)})=f^\sigma(S(p_n^{(k)}))=g(q_n^{(k)}) \quad (k=1,2).$$

Hence by making $n\to\infty$, we have

$$f(p)=g(q^{(k)})=k \quad (k=1,2),$$

which is a contradiction. Similarly T is continuous.

4. Completion of the proof. By Lemmas 3, 5, and 6, there exists a homeomorphism S of R_1 onto R_2 such that

$$f(p)=f^\sigma(S(p)) \quad (\text{resp. } \overline{f(p)}=f^\sigma(S(p)))$$

for any f in $A(R_1)$ and p in R_1. Let g possess a simple zero at $S(p)$[4] and $f=g^\tau$. Let V be a simply connected open neighborhood of $S(p)$ in which g is univalent. By Lemma 6, there exists a simply connected open neighborhood U of p such that

$$S(U)\subset V \quad \text{and} \quad f(U)\subset g(V).$$

Then

$$S=g^{-1}\circ f \quad (\text{resp. } S=g^{-1}\circ\bar{f})$$

is a local representation of S in U, which shows that S is a direct (resp. indirect) conformal transformation of R_1 onto R_2.

Finally, we show the unicity of S. Suppose that S' is a direct (resp. indirect) conformal transformation of R_1 onto R_2 such that

$$f=f\circ S' \quad \text{(resp. } \bar{f}=f\circ S'\text{)}$$

for any f in $A(R_1)$. Then for any f in $A(R_1)$,

$$f^{\sigma}(S(p))=f^{\sigma}(S'(p))$$

on R. If $S\neq S'$, then there exists a point p in R_1 such that

$$q=S(p)\neq S'(p)=q'.$$

By Florack's theorem,[4)] there exists a function g in $A(R_2)$ such that $g(q)\neq g(q')$. Set $f=g^{\tau}$. Then f belongs to $A(R_1)$ and

$$f^{\sigma}(S'(p))=f^{\sigma}(S(p))=g(q)\neq g(q')=f^{\sigma}(S'(p)),$$

which is clearly a contradiction.