17. Evans' Harmonic Functions on Riemann Surfaces

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1. Besides its own interest, Evans' harmonic function on open Riemann surfaces is important in the function theory on open Riemann surfaces. In this note, we shall sketch a method to construct Evans' harmonic function on open Riemann surfaces. The detail will be published elsewhere.

DEFINITION (*Boboc-Constantinescu-Cornea* [1]). *Let R be a hyperbolic Riemann surface and $\mathfrak{F}$ be the class of all sequences $(z_n)_{n\geq 1}$ of points in R which do not accumulate in R and*

$$\liminf_{n\to\infty} g(z_n, z_0)>0,$$

where $g(z, z_0)$ is Green's function on R with its pole z_0 in R. An Evan's function $S(z)$ on R is a positive continuous superharmonic function on R such that

$$\lim_{n\to\infty} S(z_n)=\infty$$

for any (z_n) in $\mathfrak{F}$. Moreover if $S(z)$ is harmonic on R, we call $S(z)$ an Evans' harmonic function on R.

Boboc, Constantinescu and Cornea [1] proved the existence of Evans' function on R. In the case where $R=R'-\overline{R}'_0$, where R' is a parabolic Riemann surface and R'_0 is a relatively compact subdomain of R' with smooth boundary, Kuramochi [2] proved the existence of Evans' harmonic function on R, from which the existence of Evans-Selberg's potential on R' follows at once by using the linear operator method of Sario [8]. The present author [6] gave an alternating proof of Kuramochi's result. Here we state the following

THEOREM. *There exists an Evan's harmonic function on hyperbolic Riemann surfaces.*

2. For the proof of our theorem, we use the theory be Royden's compactification. The present method to construct the desired function is already used partly in [6] and [7].

Let R be an arbitrary Riemann surface and $M(R)$ be the *Royden's algebra* associated with R, i.e. the algebra of all complex-valued absolutely continuous functions in the sense of Tonelli which are bounded and of finite Dirichlet integral. The algebra $M(R)$ is a Banach algebra with the norm $||f||=\sup(|f(z)|; z\in R)+\sqrt{D(f)}$ and the subalgebra $M(R)\cap C^\infty(R)$ is dense in $M(R)$ with respect to this norm. Hence Green's formula and the Dirichlet principle can be freely applied to functions in $M(R)$ ([3]).

Let R^* be *Royden's compactification* of R, i.e. the compact Hausdorff space containing R as its open and dense subspace and every function in $M(R)$ can be continuously extended to R^* and $M(R)$ separates points in R^*. In other words, R^* is the structure space (or character space) of the Banach algebra $M(R)$ ([3]).

A sequence $(f_n)_{n\geq 1}$ of functions on R is said to converge to a function f on R in B- (or D-) *topology* if (f_n) is bounded and converges to f uniformly on every compact subset of R (or if $\lim_{n\to\infty} D(f_n - f) = 0$). If (f_n) converges to f in B- and D-topology, then we say that (f_n) converges to f in *BD-topology*. The algebra $M(R)$ is complete with respect to the BD-topology ([3]).

Royden's boundary Γ of R is the set R^*-R. The totality of regular points in Γ with respect to the Dirichlet problem is denoted by Δ. This set Δ is characterized by

$$\Delta = (p; p \in R^*, f(p) = 0 \text{ for any } f \text{ in } M_\Delta(R)),$$

where $M_\Delta(R)$ is the BD-closure of the subalgebra $M_0(R)$ of all the functions in $M(R)$ with compact supports (see [3] and [5]).

Although some of them need a slight modification, the proofs of the following facts are found in [4].

LEMMA 1. *Royden's algebra $M(R)$, considered as a subspace of the algebra $C(R^*)$ of all complex-valued bounded continuous functions on R^*, is dense in $C(R^*)$ with respect to the norm $\|f\|_\infty = \sup(|f(z)|; z \in R)$.*

LEMMA 2. *The totality of real-valued functions in $M(R)$ forms a vector lattice with lattice operations $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$.*

LEMMA 3 (*Harmonic Decomposition*). *Let K be a compact set in R^* such that if the relative boundary $\partial(K \cap R)$ is not empty, then it consists of free analytic Jordan arcs. Then every function f in $M(R)$ can be uniquely decomposed into the form $f = u + g$, where $u \in HD(R-K)$ and $u = f$ on the set $\overline{(K \cap R)} \cup \Delta$ (here we promise that $HD(R-K) = (0)$ if $R-K$ is parabolic). Moreover $\|u\|_\infty \leqq \|f\|_\infty$ and $D(u, g) = 0$.*

LEMMA 4 (*Maximum Principle*). *Let U be an open set in R such that $\overline{U} \cap \Delta = \phi$. Then $U \in SO_{HB}$, i.e. there exists no non-zero bounded harmonic function in U vanishing on ∂U except for a polar set in ∂U.*

3. As a corollary of Harnack's inequality, we get

LEMMA 5. *Let $u(z, p)$ be a non-negative function defined on $R \times R^*$ such that $u(z, p)$ is harmonic in $z \in (R-(p))$ for fixed $p \in R$ and $u(z, p)$ is continuous in $p \in R^*$ and finitely continuous in $p \in (R^*-(z))$ for fixed $z \in R$. Then $u(z, p)$ is continuous on $R \times R^*$ and harmonic in $z \in (R-(p))$ for fixed $p \in R^*$.*

A non-negative real-valued function f on R is said to be *quasi-*

Dirichlet finite if $D(\min(f, c)) < \infty$ for every $c > 0$. Then we have

LEMMA 6. *If f is quasi-Dirichlet finite on R, then f can be uniquely extended to R^* so as to be continuous on R^*.*

By Lemmas 5 and 6, we get

LEMMA 7. *Let $g(z, w)$ be Green's function on R. Then $g(z, w)$ can be considered to be continuous in $w \in R^*$ and finitely continuous in $w \in (R^* - (z))$ for fixed $z \in R$ and $g(z, p)$ is a quasi-Dirichlet finite harmonic function in $z \in (R - (p))$ for fixed $p \in R^*$.*

By this lemma, we can define

$$G(p, q) = \lim_{R \ni z \to p} (\lim_{R \ni w \to q} g(z, w))$$

for $(p, q) \in R^* \times R^*$, which we call the *Green kernel* on R^*. Then we get

PROPOSITION 1. *The Green kernel satisfies the following conditions:*

(a) $G(z, w) = g(z, w)$ *on* $R \times R$;

(b) $G(z, p) = G(p, z)$ *if z is in R*;

(c) $G(z, p)$ *is harmonic in $z \in R$ except p*;

(d) $G(p, q)$ *is continuous in $p \in R^*$ for fixed $q \in R^*$*;

(e) $G(z, p)$ *is continuous on $R \times R^*$ and finitely continuous on $R \times \Gamma$.*

4. Let Ω be an arbitrary set which contains at least one point and K be a mapping of $\Omega \times \Omega$ into $[c, \infty]$, where $c > -\infty$. For each non-empty subset $X \subset \Omega$, we set

$$\binom{n}{2} D_n^*(X) = \inf_{p_1, \cdots, p_n \in X} \sum_{i<j}^{1, \cdots, n} K(p_i, p_j).$$

Since $(D_n^*(X))_{n \geq 1}$ is non-decresing and so we can define

$$D^*(X) = \lim_{n \to \infty} D_n^*(X).$$

Similarly we set

$$nE_n^*(X) = \sup_{p_1, \cdots, p_n \in X} (\inf_{p \in X} \sum_{i=1}^{n} K(p, p_i)).$$

Since $(n+m) E_{n+m}^*(X) \geq n E_n^*(X) + m E_m^*(X)$, we can define

$$E^*(X) = \lim_{n \to \infty} E_n^*(X).$$

Then we can prove the following similarly as in [6] and [7].

PROPOSITION 2. $E^*(X) \geq D^*(X)$.

Hereafter we always use notations D^* and E^* for $\Omega = R$ and $K(p, q) = G(p, q)$.

5. Let z_0 be a fixed point in the hyperbolic Riemann surface R and $(r_n)_{n \geq 1}$ be a sequence of positive numbers such that

$$r_n > r_{n+1}, \lim_{n \to \infty} r_n = 0$$

and the level curve $(z \in R; G(z, z_0) = r_n)$ consists of a countable number of analytic Jordan curves. Moreover we assume that the set

$$U_n = (z \in R; \mathrm{G}(z, z_0) > r_n)$$

is not relatively compact. Then we get

LEMMA 8. *The set U_n is a subdomain in R and $\overline{U}_n \cap \Delta = \phi$.*

We set $\Gamma_n=\overline{U}_n\cap\Gamma$. Let $(R_n)_{n\geq 1}$ be a normal exhaustion of R such that $z_0\in R_1$. We set $K_{n,m}=\overline{U}_n\cap\partial R_m$. We denote by $S_{n,m}$ the totality of unit positive Borel measures on $K_{n,m}$ and set

$$I(\mu)=\int G(z, w)d\mu(z)d\mu(w).$$

Then we have

LEMMA 9. *There exists a unique measure $\mu_{n,m}$ in $S_{n,m}$ such that $I(\mu_{n,m})=\inf(I(\mu); \mu\in S_{n,m})$ and the function $V_{n,m}(z)=\int G(z, w)\,d\mu_{n,m}(w)$ is a harmonic function on $R-K_{n,m}$ with $V_{n,m}(z)\leq I(\mu_{n,m})$ on R and $V_{n,m}(z)=I(\mu_{n,m})$ on $K_{n,m}$.*

LEMMA 10. $D^*(K_{n,m})=I(\mu_{n,m})$.

LEMMA 11. *There exists a unique harmonic function $w_{n,m}$ on $U_{n+1}-\overline{R}_m$ such that $w_{n,m}=0$ on ∂U_{n+1} and $w_{n,m}=1$ on $K_{n+1,m}$. Moreover $w_{n,m}$ is continuous on $\overline{U}_{n+1}-R_{m+1}$ and there exists a constant $\sigma_n>0$ such that*

$$w_{n,m}(p)\geq\sigma_n \quad (p\in\Gamma_n;\ m=1,2,\cdots).$$

LEMMA 12. $D^*(\Gamma_n)\geq\sigma_n^2 D^*(K_{n,m}) \quad (m=1,2,\cdots)$.

LEMMA 13. *There exists a unique continuous function $u_{n,m}$ on R^* such that $u_{n,m}=1$ on $K_{n,m}$ and $u_{n,m}=0$ on Δ and harmonic in $R-K_{n,m}$. Moreover*

$$\int_{\partial R_{m+1}} {}^*du_{n,m}=D(u_{n,m}) \text{ and } \lim_{m\to\infty} D(u_{n,m})=0.$$

LEMMA 14. $I(\mu_{n,m})u_{n,m}(z)=V_{n,m}(z)$ *on R and* $I(\mu_{n,m})=2\pi/D(u_{n,m})$.

Lemmas 8 and 13 are obtained by using lemmas in § 2. Lemmas 9 and 10 are well known in the potential theory (see for example, [1]). The function $w_{n,m}$ in Lemma 11 is easily constructed by using the function $G(z, z_0)-r_{n+1}$. Lemma 14 is a direct consequence of Lemmas 9 and 13. Lemma 12 is one of the key lemmas of our proof and obtained similarly as in [7] from Lemma 11. From Proposition 2, Lemmas 10, 12, 13, and 14, it follows that

PROPOSITION 3. $E^*(\Gamma_n)=\infty \quad (n=1,2,3,\cdots)$.

6. *Proof of Theorem.*[1] Let z_0 be a fixed point in R. By Proposition 3, using the standard method, we can construct an *HP* function $e_n(z)$ on R such that $e_n(z_0)=1$ and for any p in Γ_n,

$$\lim_{R\ni z\to p} e_n(z)=\infty$$

in the topology of R^*. Then

$$e(z)=\sum_{n=1}^{\infty} e_n(z)/2^n$$

is an *HP* function on R such that for any p in $\bigcup_{n=1}^{\infty}\Gamma_n$,

$$(*) \qquad \lim_{R\ni z\to p} e(z)=\infty$$

in the topology of R^*.

1) If $\mathfrak{F}$ is empty, then by the definition, any *HP* function on R is an Evans, harmonic function and so our theorem is trivial. Hence we only treat the case where $\mathfrak{F}$ is not empty.

It is easy to see that the intersection of Γ and the closure of the set (z_n) in $\mathfrak{F}$ is contained in $\bigcup_{n=1}^{\infty}\Gamma_n$, we get from (*) that

$$\lim_{n\to\infty} e(z_n)=\infty$$

for any (z_n) in $\mathfrak{F}$.

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