63. On a Theorem of Cluster Sets

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1. Let D be an arbitrary domain in the z-plane with boundary Γ and let E be a totally disconnected closed set contained in Γ. Supposing that $w=f(z)$ is non-constant, single-valued and meromorphic in D, we associate with each point $z_0 \in \Gamma$ the following sets of values.

(i) *The cluster set* $C_D(f, z_0)$. $\alpha \in C_D(f, z_0)$ if there exists a sequence of points $\{z_n\}$ with the following properties:

$$z_n \in D, \lim_{n\to\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to\infty} f(z_n) = \alpha.$$

(ii) *The boundary cluster set* $C_{\Gamma-E}(f, z_0)$. $\alpha \in C_{\Gamma-E}(f, z_0)$ if there exists a sequence of points $\{\zeta_n\}$ of $\Gamma-(\{z_0\}\cup E)$ such that

$$w_n \in C_D(f, \zeta_n) \quad \text{for each} \quad n,$$
$$z_0 = \lim_{n\to\infty} \zeta_n \quad \text{and} \quad \alpha = \lim_{n\to\infty} w_n.$$

(iii) *The range of values* $R_D(f, z_0)$. This is the set of values α such that

$$z_n \in D, \lim_{n\to\infty} z_n = z_0 \text{ and } f(z_n) = \alpha \text{ for every } n.$$

In the theory of cluster sets, the following theorem is one of the most important.[1]

THEOREM. *If E is of capacity*[2] *zero and z_0 is a point of E such that $U(z_0)\cap(\Gamma-E)\neq\phi$ for every neighborhood $U(z_0)$ of z_0, then the set*

$$\Omega = C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$$

is empty or open.

In the case where D is the unit disc $|z|<1$, we can replace $C_{\Gamma-E}(f, z_0)$ in this theorem by a considerably smaller set and obtain yet the same assertion (see Ohtsuka [5] and Noshiro [3]).[1] We shall show in the below that, in the general case where D is an arbitrary domain, we can also replace $C_{\Gamma-E}(f, z_0)$ by a considerably smaller set to obtain the same assertion of the theorem.

2. We now define new sets of values.

(iv) *The cross cluster set* $C_D^{\oplus}(f, z_0)$. $\alpha \in C_D^{\oplus}(f, z_0)$ if there exists a sequence of points $\{z_n\}$ with the following properties:

$$z_n \in D, \ \Re z_n = \Re z_0 \text{ or } \Im z_n = \Im z_0 \text{ for each } n,^{3)}$$
$$\lim_{n\to\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to\infty} f(z_n) = \alpha.$$

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1) Cf. Noshiro [4].

2) In this note, capacity means always logarithmic capacity.

3) For a complex number z, we denote by $\Re z$ and $\Im z$ the real and the imaginary part of z respectively.

When there is a neighborhood U of z_0 such that the parts of two lines $\Re z=\Re z_0$ and $\Im z=\Im z_0$ lying in U do not intersect D, we set $C_D^{\oplus}(f, z_0)=\phi$.

(v) *The cross boundary cluster set* $C_{\Gamma-E}^{\oplus}(f, z_0)$. This is obtained by replacing the cluster set $C_D(f, \zeta_n)$ by the cross cluster set $C_D^{\oplus}(f, \zeta_n)$ in the definition (ii) of the boundary cluster set $C_{\Gamma-E}(f, z_0)$.

Then we have the following amelioration of the theorem stated in § 1.

THEOREM 1. *If E is of capacity zero and z_0 is a point of E such that $U(z_0)\frown(\Gamma-E)\neq\phi$ for every neighborhood $U(z_0)$ of z_0, then the set*

$$\Omega^{\oplus}=C_D(f, z_0)-C_{\Gamma-E}^{\oplus}(f, z_0)$$

is empty or open.

Proof. The proof is similar to that of the original theorem. Supposing that $\Omega^{\oplus}\neq\phi$, we let w_0 be an arbitrary point belonging to $\Omega^{\oplus}$. Then there exists a square S: $|\Re z-\Re z_0|=r$, $|\Im z-\Im z_0|=r$, arbitrarily small, such that $S\frown E=\phi$ and $f(z)\neq w_0$ on $S\frown D$. We take here r so small that w_0 does not belong to the closure M_r of the union $\bigcup_{\zeta} C_D^{\oplus}(f, \zeta)$ for all ζ belonging to the intersection of $\Gamma-E$ with $\overline{(S)}$: the closure of the interior (S) of S. Let ρ' denote the distance of M_r from w_0 and ρ'' a positive number such that $|f(z)-w_0|\geqq\rho''>0$ on $S\frown D$. Let ρ be a positive number such that $0<\rho<\min\{\rho', \rho''\}$. Since w_0 is a cluster value of $w=f(z)$ at z_0, there exists a sequence of points $z_n\in(S)\frown D(n=1, 2, \cdots)$ tending to z_0 such that $w_n=f(z_n)$ tends to w_0. The inverse image D_0 of (c): $|w-w_0|<\rho$ in $(S)\frown D$ consists of at most a countable number of connected components. The component containing z_n is denoted by Δ_n (which may coincide with other $\Delta_{n'}$).

First we shall prove that, for each n, the intersection Z of the closure $\bar{\Delta}_n$ of Δ_n with Γ is a closed set of capacity zero. To prove this, it is enough to show that the set $Z-E$ is a countable set. We note that, for each point ζ of $Z-E$, there is a positive number $r(\zeta)$ such that the parts of two lines L_ζ: $\Im z=\Im\zeta$ and L'_ζ: $\Re z=\Re\zeta$ lying on $|z-\zeta|\leqq r(\zeta)$ do not intersect Δ_n. For otherwise $C_D^{\oplus}(f, \zeta)$ would contain a value belonging to $\overline{(c)}$: $|w-w_0|\leqq\rho$ and we would be led to a contradiction that M_r intersects $\overline{(c)}$. Set

$$Z_p=\left\{\zeta\in Z-E;\ r(\zeta)>\frac{1}{p}\right\},\quad (p=1, 2, \cdots).$$

Then

$$Z-E=\bigcup_{p=1}^{\infty} Z_p.$$

We now prove that each Z_p consists of only isolated points so that Z_p, consequently $Z-E$, is countable. Contrary, suppose that Z_p

contains a point ζ_0 which is not isolated. Then there exists a sequence of points $\zeta_k \in Z_p (k=1,2,\cdots)$ tending to ζ_0. It is easily seen that there exists a point z_1 in Δ_n which lies outside of the square S_{ζ_0}: $|\Re z-\Re\zeta_0|=1/p$, $|\Im z-\Im\zeta_0|=1/p$. Suppose that an infinite number of ζ_k do not lie on the lines $L_{\zeta_0}: \Im z=\Im\zeta_0$ and $L'_{\zeta_0}: \Re z=\Re\zeta_0$. Then there is a subsequence $\{\zeta_{k_j}\}$ of $\{\zeta_k\}$ tending to ζ_0 outside of $L_{\zeta_0}\cup L'_{\zeta_0}$ and there appears a pair $(\zeta_{k_j}, \zeta_{k_{j'}})$ among $\{\zeta_{k_j}\}$ such that $\zeta_{k_{j'}}$ lies inside of the rectangle R with sides contained in $L_{\zeta_0}, L'_{\zeta_0}, L_{\zeta_{k_j}}$ and $L'_{\zeta_{k_j}}$. This is impossible. For $\zeta_{k_{j'}}$ is a boundary point of Δ_n so that there is a point z' of Δ_n arbitrarily near $\zeta_{k_{j'}}$ and we can join z' and z_1 with a continuous curve in the domain Δ_n. But on the other hand this continuous curve can not intersect any side of R since ζ_0 and ζ_{k_j} are points of Z_p. Contradiction. Thus we may assume that $\Re\zeta_k=\Re\zeta_0$ and $\Im\zeta_1>\Im\zeta_2>\cdots\to\Im\zeta_0$. Since every ζ_k is a boundary point of Δ_n, we can find a point ζ'_k of Δ_n arbitrarily near ζ_k and join it and z_1 with a continuous curve Λ_k in the domain Δ_n which can not intersect $\bigcup_{k=1}^{\infty}(L_{\zeta_k}\cap(\overline{S_{\zeta_0}}))$, where we denote by $(\overline{S_{\zeta_0}})$ the closure of the interior of S_{ζ_0}. Hence a subsequence of $\{\Lambda_k\}$, $\Lambda_k\subset\Delta_n$, and the sequence of segment $L_{\zeta_k}\cap(\overline{S_{\zeta_0}})$, $(L_{\zeta_k}\cap(\overline{S_{\zeta_0}}))\cap\Delta_n=\phi$, simultaneously cluster to at least the left or the right side of ζ_0 of the segment $L_{\zeta_0}\cap(\overline{S_{\zeta_0}})$ and we see that at least one of them, we denote it by L, is contained in Z. Since E is of capacity zero, $L-E$ is not empty. Let ζ be a point of $L-E$. Then, for any $r>0$, $L'_{\zeta}\cap\{|z-\zeta|\leqq r\}$ intersects some Λ_k, consequently Δ_n; this contradicts that ζ belongs to some $Z_{p'}$. Thus Z'_p consists of isolated points and hence is countable. We can conclude that $Z-E$ is countable and is of capacity zero.

Suppose that there is an infinite number of distinct components Δ_n. In this case, we assume for simplicity that $\Delta_n\neq\Delta_m$ if $n\neq m$. Then $\Delta_n(n=1,2,\cdots)$ converges to z_0. For, if not, there is a square S': $|\Re z-\Re z_0|=r'$, $|\Im z-\Im z_0|=r'$ $(r'<r)$ such that $S'\cap E=\phi$ and $S'\cap\Delta_{n_\nu}\neq\phi(\nu=1,2,\cdots)$ for a subsequence $\{\Delta_{n_\nu}\}$ of $\{\Delta_n\}$. Let ζ_ν be a boundary point of Δ_{n_ν} on the square S' and ζ_∞ an accumulation point of the sequence $\{\zeta_\nu\}$. Clearly $f(\zeta_\nu)$ lies on the circle c: $|w-w_0|=\rho$. It is also clear that $\zeta_\infty\in(\Gamma-E)\cup D$. This leads us to a contradiction because M_r meets the circle c if $\zeta_\infty\in\Gamma-E$, or else infinitely many of the level curves: $|f(z)-w_0|=\rho$ meet a small neighborhood of ζ_∞ inside of D if $\zeta_\infty\in D$. Since for each Δ_n, $\Gamma\cap\overline{\Delta}_n$ is of capacity zero as we have seen in the above, the value-set $D_n=f(\Delta_n)$ covers (c) with possible exception of capacity zero and hence the closure $\overline{D}_n=(\overline{c})$: $|w-w_0|\leqq\rho$. Noticing that Δ_n converges to z_0, we see that $C_D(f, z_0)\supset(\overline{c})$.

Next we consider the case where there is a finite number of distinct components Δ_n. In this case, there is at least one component, say Δ_1, containing a subsequence $\{z_{n_\nu}\}$ of $\{z_n\}$, and the boundary Λ of Δ_1 satisfies the condition that $U(z_0)\frown(\Lambda-Z)\neq\phi$ for every neighborhood of z_0, where $Z=\Gamma\frown\bar{\Delta}_1$. Since Z is a closed set of capacity zero, we can take Δ_1, Λ and Z as D, Γ and E in the theorem stated in §1 respectively and have $C_{\Delta_1}(f, z_0)\supset(\bar{c})$. Obviously $C_D(f, z_0)\supset C_{\Delta_1}(f, z_0)$ and hence $C_D(f, z_0)\supset(\bar{c})$.

Thus we have in both cases $C_D(f, z_0)\supset(\bar{c})$. On the other hand, we have taken ρ so small that $M_r\frown\{|w-w_0|\leqq\rho\}=\phi$. Therefore $\Omega^{\oplus}=C_D(f, z_0)-C^{\oplus}_{\Gamma-E}(f, z_0)\supset(c)$. w_0 is an interior point of $\Omega^{\oplus}$ and $\Omega^{\oplus}$ is open because of arbitrariness of $w_0\in\Omega^{\oplus}$. Our proof is now complete.

3. By the same arguments as in the classical case, we can prove also the following theorems.

THEOREM 2. *Under the same assumption as in* Theorem 1, *$R_D(f, z_0)$ covers $\Omega^{\oplus}$ except for a possible set of capacity zero.*

THEOREM 3. *If $\Omega^{\oplus}-R_D(f, z_0)\neq\phi$ in* Theorem 2, *then each value α of $\Omega^{\oplus}-R_D(f, z_0)$ is an asymptotic value of $f(z)$ at z_0 or there is a sequence of points $\zeta_n\in\Gamma(n=1, 2,\cdots)$ tending to z_0 such that α is an asymptotic value of $f(z)$ at each ζ_n.*

THEOREM 4. *In* Theorem 2, *if each point of E is contained in a non-degenerate connected component of Γ, then $R_D(f, z_0)$ covers every connected component of the open set $\Omega^{\oplus}$ except for at most two values (an amelioration of a theorem of Hervé* [1]*).*

Remark. We can change slightly the definitions of the cross cluster set and the cross boundary cluster set to obtain the same results as above. For instance, we change, in the definiton (iv), the condition that $\Re z_n=\Re z_0$ or $\Im z_n=\Im z_0$ for each n by the condition that $|z_n|=|z_0|$ or $\arg z_n=\arg z_0$ for each n and define the cross boundary cluster set $C^{\oplus}_{\Gamma-E}(f, z_0)$ using this new cross cluster set. In the case where D is the unit disc $|z|<1$, this $C^{\oplus}_{\Gamma-E}(f, z_0)$ coincides with the boundary cluster set of Ohtsuka [5] and Lohwater [2].

References

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