61. A Remark on Gentzen's Paper "Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie". II

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In this paper we shall define systems $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and prove the theorem stated in the first paper of this title for these systems.

Definition of the system $\mathfrak{S}_2$. $\mathfrak{S}_2$ is a system obtained from G^1LC modifying it as follows (cf. [8]):

1. Every beginning sequence of $\mathfrak{S}_2$ is of the form $D\to D$ or of the form $a=b$, $A(a)\to A(b)$ or a 'mathematische Grundsequenz' in the sense of the first paper.

2. The inference-schema 'induction' is added.

3. The inference V left on an f-variable of the form

$$\frac{F(V),\ \Gamma\to\Delta}{V\varphi F(\varphi),\ \Gamma\to\Delta}$$

is restricted by the condition that $V\varphi F(\varphi)$ is f-closed, i.e. $V\varphi F(\varphi)$ does not contain any free f-variable.

The proof of the theorem and the result (†) for $\mathfrak{S}_2$ can be performed in the same way as for $\mathfrak{S}_1$.

The definition of $\mathfrak{S}_3$. Let $I(a)$ and $a<^*b$ be two primitive recursive predicates. Let us assume that the following condition is satisfied: $<^*$ is a well-ordering of I, where I is $\{a\,|\,I(a)\}$.

Now the formal system $\mathfrak{S}_3$ is obtained as follows from G^1LC.

1. Every beginning sequence is of the form $D\to D$ or of the form $a=b$, $A(a)\to A(b)$ or the 'mathematische Grundsequenz' in the sense of the first paper or the following form.

$$I(a),\ A_j(a,b)\to G_j(a,b\{x,y\}(A_j(x,y)\wedge x<^*a))$$

(*)

$$I(a),\ G_j(a,b,\{x,y\}(A_j(x,y)\wedge x<^*a))\to A_j(a,b)\qquad j=0,1,2,\cdots.$$

Here $\{x,y\}$ is used instead of usual notations $\hat{x}\hat{y}$, λxy and $A_1, A_2, A_3,\cdots$ are new symbols for predicates. Moreover, $G_j(j=0,1,2,\cdots)$ are arbitrary formulas satisfying the following conditions:

a) $G_j(a,b,\alpha)$ does not contain $A_j, A_{j+1}, A_{j+2},\cdots$.

b) If $G_j(a,b,\alpha)$ contains the figures of the form $V\varphi A(\varphi)$, then $A(\beta)$ does not contain any bound f-variable.

2. The inference-schema called 'induction' is added.

3. The inference V left on an f-variaqle of the form

$$\frac{F(V),\ \Gamma \rightarrow \Delta}{V\varphi F(\varphi),\ \Gamma \rightarrow \Delta}$$

is restricted by the condition that $F(\alpha)$ does not contain any bound f-variable. It should be remarked that $F(\alpha)$ may contain $A_0, A_1, A_2, \cdots$ and V may contain bound f-variables and $A_0, A_1, A_2, \cdots$.

The consistency of this system is proved by using the transfinite induction of a system $O(\{\infty_1, \infty_2\} \smile \hat{I}, \hat{I})$ of ordinal diagrams (cf. [**9**]). To define the reduction for a TJ_3-proof-figure whose end number is not 0, we follow the consistency proof of $\mathfrak{S}_3$. We assign the same ordinal diagram to every sequence of a proof-figure of $\mathfrak{S}_3$ as in [**9**] and assign

$$(\infty_2, 0, (\infty_2, 0, (\infty_2, 0, (\infty_2, 0 \# (\infty_2, 0, 0)))))$$

where 0 stands for the first element of $\hat{I}$.

The proof can be performed similarly as for $\mathfrak{S}_1$ reading p 1–4 and 5 there as p 1–4 and 5′.

p 5′. The end-place of $\mathfrak{P}$ contains no beginning sequence of the form (∗) (for inductive definition).

It remains open if any well-ordering whose order-type is less than that of the well-ordering on $O(\{\infty_1, \infty_2\} \smile \hat{I}, \hat{I})$ is provable in $\mathfrak{S}_3$.

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