99. On the Sonnenschein Methods of Summability

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The class of summability methods introduced by Sonnenschein [5] is of relatively recent origin. The Sonnenschein methods, whose definition follows, include in their collection the Euler methods (E, p) from among the Hausdorff methods, the Hardy-Littlewood-Fekete circle methods (T, α) which are quasi-Hausdorff methods and also the Laurent series methods (S, α) of Meyer-König [2] and Vermes [6]. Trivially they include also the identity method, which is a member of each of the classes of Hausdorff, quasi-Hausdorff, and Nörlund methods as also of the class of (S^*, μ) methods of the present author [3] and of the Karamata methods $K[\alpha, \beta]$. (For the definition of the Karamata methods, see, for instance Sledd [4]). We show here that the Sonnenschein methods have no other methods in common with any of the five classes mentioned above. The catalyst for the enquiry is the result of Agnew [1] that the Cesàro methods are the only methods of summability, regular or not, which are both Nörlund methods and Hausdorff methods.

Brief definitions of the various methods follow. All the methods we consider are matrix methods, provided by matrices of the type $A=(a_{nk})$, n denoting the row-index and k, the column index.

The Sonnenschein methods are defined by matrices $F=(f_{nk})$, related to a function $f(z)$, regular in $|z|<R, R\geq 1, f(1)=1$ and the elements of the matrix are given by

$$[f(z)]^n=\sum_{k=0}^{\infty} f_{nk}z^k, \text{ for each } n, \text{ with } f_{00}=1 \text{ and } f_{0k}=0, k\neq 0.$$

The Hausdorff method (H, μ) is defined by the matrix $H=(h_{nk})$ where

$$h_{nk}=\binom{n}{k}\Delta^{n-k}\mu_k, \ (n\geq k), \text{ and } =0 \ (n<k).$$

The quasi-Hausdorff method (H^*, μ) is given by the matrix $H^*=(h^*_{nk})$ where

$$h^*_{nk}=0, \ (n>k) \text{ and } =\binom{k}{n}\Delta^{k-n}\mu_n, \ (n\leq k).$$

The (S^*, μ) methods are given by the matrix $S^*=(s^*_{nk})$, with

$$s^*_{nk}=\binom{n+k}{k}\Delta^n\mu_k \text{ for all } n \text{ and } k.$$

The Euler method (E, p), the circle method (T, p), the Laurent

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series method (S, p) correspond respectively to the Hausdorff, quasi-Hausdorff and (S^*, μ) methods with $\mu_n = p^n$.

The Nörlund method (N, p_n) is provided by the matrix $A=(a_{nk})$, with $a_{nk}=p_{n-k}|P_n, (n\geq k)$ and $=0, (n<k)$, where p_n is a sequence of real or complex constants and $P_n=p_0+p_1+\cdots+p_n$, for each n. The class of Nörlund methods is identical with the class of triangular methods defined by the matrices $B=(b_{nk})$ such that $b_{nn}\neq 0$, $\sum_{k=0}^{n} b_{nk}=1$ for each n and for each $k=0,1,2,\cdots$ there exists a constant c_k such that $b_{n,n-k}=c_k b_{nn}$, $n\geq k$. (For details, see Agnew [1]). It is in this form we shall refer to the Nörlund methods in this paper. The choice $p_0=1$ and $p_n=0$ $(n\neq 0)$ reduces the Nörlund method (N, p_n) to the identity method.

Proposition 1. *The identity transformation is the only method, regular or not, that is both Sonnenschein and Nörlund.*

Proof. Let the method defined by the matrix $F=(f_{nk})$ be both Nörlund and Sonnenschein. Since F is assumed to be Nörlund, it may be taken as the matrix for which $f_{nn}\neq 0$, $\sum_{k=0}^{n} f_{nk}=1$ for each n and for each k there exists a c_k such that $f_{n,n-k}=c_k f_{nn}$, $(n\geq k)$. Since F is also assumed to be Sonnenschein, we have, corresponding to a suitably defined function $f(z)$, regular in $|z|<R$, $R\geq 1$,

$$(1)\qquad \begin{aligned}[][f(z)]^n &= \sum_{k=0}^{n} f_{nk} z^k = \sum_{k=0}^{n} f_{n,n-(n-k)} z^k = f_{nn}\sum_{k=0}^{n} c_{n-k} z^k \\ &= f_{nn}[c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n].\end{aligned}$$

The above equation is valid for each n. Putting $n=0$, we get $f_{00}c_0 =1$; F being Nörlund, we have $f_{00}=1$; and it follows therefore that $c_0=1$. Consider now the equation $[f(z)]^{n+1}=[f(z)]^n\cdot[f(z)]$. Within the circle of convergence $|z|<R$, we can multiply the various power series involved and equate coefficients of like powers. Thus for $|z|<R$, we have

$$(2)\qquad \begin{cases} f_{n+1,n+1}[c_0 z^{n+1} + c_1 z^n + \cdots + c_n z + c_{n+1}] \\ \qquad = f_{nn}[c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n]\cdot f_{11}[c_0 z + c_1]. \end{cases}$$

Equating the coefficients of like powers on both sides, we obtain,

$$f_{n+1,n+1}\cdot c_0 = f_{nn}\cdot f_{11}\cdot c_0^2\,;\ \text{since}\, f_{mm}\neq 0,\ f_{n+1,n+1}=f_{nn}\cdot f_{11}\,;$$

and

$$f_{n+1,n+1}c_1 = f_{nn} f_{11}[c_1 c_0 + c_0 c_1] = 2c_1 f_{nn} f_{11}.$$

Thus $c_1=2c_1$, since $f_{n+1,n+1}=f_{nn}f_{11}$ and therefore $c_1=0$. Considering the constant terms on both sides of equation (2), we have, $c_{n+1}f_{n+1,n+1} =f_{nn}\cdot f_{11}\cdot c_n\cdot c_1$ and consequently $c_{n+1}=0$. Thus, for each $n=0,1,2,\cdots$ we have $c_{n+1}=0$ and $c_0=1$. This shows that the matrix F is the identity matrix.

Proposition 2. *The circle method (T, μ_1) is the only method, regular or not, which is both quasi-Hausdorff and Sonnenschein.*

Proof. Let F be a Sonnenschein method (corresponding to the function $f(z)$ as before). Let F be also quasi-Hausdorff. Then we have

$$f(z)=\sum_k f_{1k}z^k=\sum_{k=1}^{\infty}\binom{k}{1}\Delta^{k-1}\mu_1 z^k$$

and

$$[f(z)]^n=\sum_k f_{nk}z^k=\sum_{k=n}^{\infty}\binom{k}{n}\Delta^{k-n}\mu_n z^k.$$

Considering, as before the equation $[f(z)]^{n+1}=[f(z)]^n\cdot[f(z)]$, we get $\left(\sum_{k=1}^{\infty}\binom{k}{1}\Delta^{k-1}\mu_1 z^k\right)\left(\sum_{k=n}^{\infty}\binom{k}{n}\Delta^{k-n}\mu_n z^k\right)=\left(\sum_{k=n+1}^{\infty}\binom{k}{n+1}\Delta^{k-n-1}\mu_{n+1}z^k\right)$. Equating, as we may, coefficients of like powers of z on both sides of the above equation, we get (considering the term z^{n+1})

$$\mu_1\mu_n=\mu_{n+1}.$$

This is true for $n=0,1,2,\cdots$ and consequently $\mu_1\cdot\mu_0=\mu_1$ i.e. $\mu_0=1$. Also $\mu_1\cdot\mu_1=\mu_2$ or $\mu_2=\mu_1^2$, and inductively $\mu_n=\mu_1^n$. Thus the quasi-Hausdorff method is the method (T,μ_1).

In a similar manner one may prove also that if a method is both Sonnenschein and Hausdorff (respectively (s^*,μ)), then it is the Euler method (E,μ_1) (respectively the method (s,μ_1)). We remark also that throughout this paper "Sonnenschein methods" could be replaced by "Karamata methods", since the Karamata methods are special types of Sonnenschein methods.

References

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