95. A Classification of Orientable Surfaces in 4-Space

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Things will be considered only from the *piecewise-linear* (or semi-linear) and *combinatorial* point of view. Terminology relies heavily on [4].

Let M_i be a closed (orientable) oriented surface in an (orientable) oriented 4-manifold W_i without boundary, $i=1,2$. Then M_1 is *iso-neighboring* to M_2 if there are a regular neighborhood U_i of M_i in W_i and an onto, orientation preserving homeomorphism $\psi: U_1 \rightarrow U_2$ such that $\psi(M_1)=M_2$ where $\psi | M_1$ is orientation preserving and where the orientation of U_i is induced from W_i.

By Theorem 1 of [4], the iso-neighboring relation is an equivalence relation, and the *collection of singularities* of surface settled by [3] is an invariance under the iso-neighboring relation.

Another invariance may be defined as follows. Let a closed oriented surface M be in an oriented 4-manifold W without boundary, and let K and L be simplicial subdivisions of M and W respectively such that K is a subcomplex of L, where it is assumed without loss of generality that for each (closed) simplex of L the intersection of the simplex and M is either empty or a simplex of K.

For each vertex Δ of K, ∇ and $\square$ denote the 2-, 4-cells dual to Δ in K and L respectively. Then $\partial\nabla$ and $\partial\square$ are a circle and a 3-sphere respectively such that $\partial\nabla \subset \partial\square$, where ∂X denotes the boundary of X. Then the sum U of all 3-cells dual to 1-simplices (of K), incident to Δ, in L is a regular neighborhood of $\partial\nabla$ in $\partial\square$ by [4], whose boundary is a torus T. If orientations of $\partial\nabla$ and $\partial\square$ are induced from the orientation of ∇ and $\square$ which are naturally induced from M and W respectively, then the oriented pair $\partial\nabla, \partial\square$ may be regarded as a knot. Then, by [2], the meridian a and the longitude b are defined for the knot (where a and b are 1-cycles on T). Let Δ_0 be a fixed vertex of K. Then the cycle $\sum_j b_j$ is homologous to $w\ a_0$ in $\bigcup_j T_j$ for some integer w where j varies on vertices Δ_j of K. It is proved that the integer w, called the *Stiefel-Whitney number*, is an invariance of M in W under the iso-neighboring relation. The proof is carried out by the elementary routine of algebraic topology; w is independent of choice of Δ_0, and of subdivisions K, L concerned, so that it is invariant. A simple proof will be supplied in the subsequent paper by R. Takase [6].

Then the (dual) skelton-wise extension scheme of homeomorphism described in [4] and the argument in [1] furnish the proof of the main result;

Theorem A. *Let M_i be a closed oriented surface in an oriented 4-manifold W_i without boundary, $i=1, 2$, such that M_1 and M_2 are homeomorphic. Then M_1 and M_2 are iso-neighboring if and only if they have same collection of singularities and same Stiefel-Whitney number.*

By the argument due to [5], it is shown that $w=0$ if M is in (euclidean) 4-space. Therefore

Corollary to Theorem A. *Let M_1 and M_2 be closed oriented surfaces in 4-space such that M_1 and M_2 are homeomorphic. Then M_1 and M_2 are iso-neighboring if and only if they have same collection of singularities.*

A closed orientable surface M may be imbedded in a 3-space, and then whose regular neighborhood in a 4-space containing the 3-space is the product of M and a 2-cell. Hence

Theorem B. *If a closed surface M in 4-space R is locally flat (=no singular point) then the boundary of regular neighborhood of M in R is the product of M and a circle.*

Theorem B may be false if M is not locally flat.

References

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