126. On a Characteristic Property of Confocal Conic Sections

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1963)

In this paper we shall characterize confocal conic sections from the standpoint of conformal mapping by an entire function. In the previous papers (see [1], [2]) we discussed conic sections in detail from the same standpoint making Ivory's Theorem the principal subject.

From the fact that the mapping by a non-constant entire function $w=f(z)$ is conformal we can conclude that the horizontal and vertical lines $\mathrm{Im}(z)=\mathrm{const.}$ and $\mathrm{Re}(z)=\mathrm{const.}$ are transformed by the function into the two families of curves which intersect each other at right angles. Then, we denote an arbitrary curvilinear rectangle by $C_1 C_2 C_3 C_4$ where C_1, C_2, C_3, and C_4 are four complex constants.

Theorem. If γ is a fixed point in the w-plane and if $|\gamma-C_1|+|\gamma-C_3|=|\gamma-C_2|+|\gamma-C_4|$, then the two families of curves above are confocal conic sections which have their common foci at the point γ.

Proof. By hypothesis we have the following functional equation:

(1) $$|f(x+y)-\gamma|+|f(x-y)-\gamma|=|f(x+\bar{y})-\gamma|+|f(x-\bar{y})-\gamma|,$$

where x, y are arbitrary complex numbers.

Putting $g(z)=f(z)-\gamma$, we have

$$|g(x+y)|+|g(x-y)|=|g(x+\bar{y})|+|g(x-\bar{y})|.$$

Putting $y=x=\frac{z}{2}=\frac{s+it}{2}$ where s, t are real and $g(z)=u+iv$ where u, v are real, we have

(2) $$\sqrt{u^2+v^2}+|g(o)|=|g(s)|+|g(it)|.$$

Differentiating (2) with respect to x and next with respect to y and using the Cauchy-Riemann equations, we have

(3) $$(-uv_{ss}+vu_{ss})(u^2+v^2)=(uu_s+vv_s)(-uv_s+vu_s).$$

Since $g(z)$ is not a constant, there exists a properly chosen domain D where $g(z)\neq 0$.

By (3) we have in D

$$\mathrm{Im}\left(\frac{2gg''-g'^2}{g^2}\right)=\mathrm{Im}\left\{\frac{2(u+iv)(u_{ss}+iv_{ss})-(u_s+iv_s)^2}{(u+iv)^2}\right\}=0.$$

Hence we have

$$\frac{2gg''-g'^2}{g^2}=A,$$

where A is a real constant.

Solving this differential equation, we have

$$f(z)=(az+b)^2+\gamma,$$
$$\text{or } f(z)=(a\cos\alpha z+b\sin\alpha z)^2+\gamma,$$
$$\text{or } f(z)=(a\cosh\alpha z+b\sinh\alpha z)^2+\gamma,$$

where a, b are complex constants and α is a real constant.

Thus the proof is completed.

References

[1] H. Haruki: On Ivory's Theorem, Mathematica Japonicae, **1**(4), 151 (1949).

[2] H. Haruki: On the Conformal Mapping by the Elementary Functions, Science Reports North College, Osaka University, no. 6, pp. 5–10 (1957).