146. *Eigenfunction Expansion Associated with the Operator $-\Delta$ in the Exterior Domain*

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(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1963)

1. Introduction. An attempt to use distorted plane waves for expanding an arbitrary function which is square integrable was first carried through by Ikebe in 1960 [1]. He treated the Schroedinger operator $-\Delta+q(x)$ in the whole 3-dimensional Euclidean space E, where Δ denotes Laplacian and $q(x)$ is a potential function. In the present paper we consider the similar problem for the Schroedinger operator of another type, i.e., of a rigid body. This means that no potential exists, but negative Laplacian has a boundary condition on some bounded, smooth and closed surface representing the rigid body. Naturally the space with which we are concerned is not the whole 3-dimensional Euclidean space but the exterior domain of the surface. The method used is essentially the same to the Ikebe's one, except for the use of the potential theory which seems indispensable in our case. No explicit mention is made of the smoothness of the surface, for it is rather complicated. The reader will find it in any textbook on the potential theory.[1]

The author would like to express his hearty thanks to Professor M. Yamaguti for many valuable suggestions and to Professor S. Mizohata by whom he was inspired the existence of the problem.

2. Exterior Dirichlet problem. S denotes a sufficiently smooth, closed and bounded surface in E. Ω is the exterior domain relative to S. Suppose that $u(x)$ satisfies

$$(2.1)\qquad (-\Delta-\kappa^2)u(x)=0, \qquad x\in\Omega,$$

$$(2.2)\qquad u(p)=f(p), \qquad p\in S,$$

$$(2.3)\qquad \frac{\partial u}{\partial r}-i\kappa u=e^{-\beta r}o(r^{-1}),$$
$$u(x)=e^{-\beta r}O(r^{-1}),\ \beta=Im\ \kappa.^{2)}$$

Then the function is called the solution of the exterior Dirichlet problem for the boundary condition $f(p)$. Then we can show the

1) See e.g. [8]. We must also suppose, as is usually supposed in the potential theory, for the surface that the following inequality holds.

$$\int\frac{|\cos(n_p,\overrightarrow{xp})|}{|x-p|^2}dS_p\leq C.$$

Here C denotes a positive constant independent of x.

2) Sommerfeld's radiation condition and finiteness condition in the generalized form. See [3].

following:

Theorem 1. *For arbitrary given continuous function $f(p)$ defined on S, there exists a unique solution of the exterior Dirichlet problem* (2.1), (2.2), (2.3), *provided that* $Im\,\kappa \geq 0$.

Along the usual line of potential theory we suppose at first that the solution can be written in the form of double layer potential.

$$u(x)=\int_S \rho(q)\frac{d}{dn_q}\left(\frac{e^{i\kappa|x-q|}}{2\pi|x-q|}\right)dS_q. \tag{2.4}$$

Here $\rho(q)$ is unknown density on S, which satisfies the following integral equation.

$$f(p)=-\rho(p)+\int_S \rho(q)\frac{d}{dn_q}\left(\frac{e^{i\kappa|p-q|}}{2\pi|p-q|}\right)dS_q. \tag{2.5}$$

Let B be the Banach space of all continuous functions defined on S. The kernel in the right member of (2.5) defines a bounded operator in B. Moreover it turns out to be a completely continuous operator. Let us denote it by T_κ. Now we can make use of the Riesz-Schauder theory.

Lemma 1. *Let* $Im\,\kappa \geq 0$. *Then* $I-T_\kappa$ *is not invertible, if and only if* κ^2 *is an eigenvalue of interior Neumann problem of* $-\Delta$.

Thus the existence of the solution is evident for the case of $\kappa^2 \notin \{\nu_n\}$, where $\{\nu_n\}$ denotes the set of all eigenvalues. For the proof of this lemma, see [2], [8].

It is well known that

$$0=\nu_0<\nu_1<\nu_2<\cdots\to\infty.$$

For the case of $\kappa^2 \in \{\nu_n\}$, Kupradze constructed the solution as a sum of simple layer and double layer potential. See [3]. But we shall not use the fact in the proof of Theorem 2. Uniqueness of the solution is guaranteed by Rellich's theorem [4]. See also [3], [5]. The following lemma will be needed later.

Lemma 2. *Let* $Im\,\kappa \geq 0$ *and* $\kappa^2 \notin \{\nu_n\}$. *Then* $(I-T_\kappa)^{-1}$ *depends continuously on* κ *in the sense of operator norm.*

3. Green function. Let H be the unique selfadjoint extension in $L_2(\Omega)$ of $-\Delta$ with zero boundary condition on S. It is well known that H has no eigenvalue and that its spectrum coincides with the full interval $[0, \infty)$. The resolvents of H are integral operators. We shall freely use the fact that for arbitrary κ ($Im\,\kappa>0$) the Green function of the exterior Dirichlet problem (2.1), (2.2), (2.3) is the resolvent kernel of H. For the existence and the property of the Green function, see [2]. It is also proved that the Green function is symmetric.

4. Distorted plane wave. Let us introduce $v(x, k; \kappa)$ and $w(x, k; \kappa)$. The former is the solution of the extrior Dirichlet problem (2.1), (2.2), (2.3) with $f(x)=-e^{ik\cdot x}$. The latter is defined as follows.

$$w(x, k; \kappa)=e^{ik\cdot x}+v(x, k; \kappa). \tag{4.1}$$

Then $v(x, k)$ and $\varphi(x, k)$ are defined with the aid of $v(x, k; \kappa)$ and $w(x, k; \kappa)$, namely

$$v(x, k)=v(x, k; |k|), \tag{4.2}$$

$$\varphi(x, k)=w(x, k; |k|)=e^{ik\cdot x}+v(x, k). \tag{4.3}$$

Obviously $\varphi(x, k)$ satisfies (2.1) with $\kappa=|k|$, (2.2) with $f(p)=0$. It represents the distorted plane wave, in other words, it describes a physical phenomenon, i.e., diffraction of a plane wave by the rigid body S. We can regard it as an eigenfunction of H.[3)]

5. Expansion theorem. We shall denote by M the 3-dimensional Euclidean space formed by all wave vectors k. Then we have an expansion theorem.

Theorem 2. (i) *There exists a transformation Z from $L_2(\Omega)$ into $L_2(M)$, such that*

$$(Zf)(k)=(2\pi)^{-\frac{3}{2}}\,\text{l.i.m.}\int_\Omega \overline{\varphi(x, k)}f(x)dx. \tag{5.1}$$

(ii) *There exists a transformation Z' from $L_2(M)$ into $L_2(\Omega)$, such that*

$$(Z'g)(x)=(2\pi)^{-\frac{3}{2}}\,\text{l.i.m.}\int_M \varphi(x, k)\, g(k)dk. \tag{5.2}$$

(iii) *Putting $Zf=\hat{f}$, the following formula holds.*

$$f(x)=(2\pi)^{-\frac{3}{2}}\,\text{l.i.m.}\int_M \varphi(x, k)\, \hat{f}(k)dk. \tag{5.3}$$

(iv) *The transformation Z is isometric.*

$$\int_\Omega |f(x)|^2\, dx=\int_M |\hat{f}(k)|^2\, dk. \tag{5.4}$$

$$\int_\Omega f(x)\overline{g(x)}\, dx=\int_M \hat{f}(k)\, \overline{\hat{g}(x)}\, dk. \tag{5.5}$$

(v) *If $f(x)$ belongs to the definition domain of H, then*

$$(Hf)(x)=(2\pi)^{-\frac{3}{2}}\,\text{l.i.m.}\int_M |k|^2\, \varphi(x, k)\hat{f}(k)dk. \tag{5.6}$$

It should be noted that this theorem only asserts the isometry of Z. The result is unsatisfactory in this respect, because the unitary equivalence of H and H_0 (self-adjoint extension of $-\Delta$ in $L_2(E)$) [6], [7] suggests the unitarity of Z.

6. Sketch of proof. $H(x, y; \kappa)$ ($Im\,\kappa>0$) denotes the Green function of the exterior Dirichlet problem (2.1), (2.2), (2.3). It is evident that $H(x,\cdot\,; \kappa)$ belongs to $L_1(\Omega)\cap L_2(\Omega)$, therefore its conjugate Fourier transform exists. Put

$$u(x, k; \kappa)=(2\pi)^{-\frac{3}{2}}\int_\Omega H(x, y; \kappa)e^{ik\cdot y}dy. \tag{6.1}$$

3) For the term "eigenfunction", see [1].

In view of definitions in 4, the following fundamental relation holds, provided that $Im\,\kappa>0$.

(6.2) $$w(x, k; \kappa)=(2\pi)^{\frac{3}{2}}(|k|^2-\kappa^2)u(x, k; \kappa)=e^{ik\cdot x}+v(x, k; \kappa).$$

Now we shall define the transformation Z along the line given by Ikebe [1]. The Parseval's equality combining with (6.2) leads to

(6.3) $$\int_\Omega H(z, x; \kappa)\overline{H(z, y; \kappa)}dz = \frac{1}{(2\pi)^3(\kappa^2-\bar{\kappa}^2)}\int_M\left\{\frac{1}{|k|^2-\kappa^2}-\frac{1}{|k|^2-\bar{\kappa}^2}\right\}w(x, k; \kappa)\overline{w(y, k; \kappa)}dk.$$

Here the symmetry of the Green function is taken into account. Multiply both sides of (6.3) by $\overline{g(x)}f(y)$, where both $f(x)$ and $g(x)$ belong to $C_0(\Omega)$. Integrate over $\Omega\times\Omega$ with respect to x and y. After interchanging the order of the integration, integrate with respect to $\mu(=\mathrm{Re}\,\kappa^2)$ from α to β $(\nu_{n-1}<\alpha<\beta<\nu_n)$. Let $\varepsilon\to 0$ $(\varepsilon=Im\,\kappa^2)$. Thus we have

(6.4) $$(E_\beta f, g)-(E_\alpha f, g)=\int_{\sqrt{\alpha}<|k|<\sqrt{\beta}}\hat{f}(k)\overline{\hat{g}(k)}dk,$$

where E_λ denotes the spectral resolution of H, and

(6.5) $$\hat{f}(k)=(2\pi)^{-\frac{3}{2}}\int_\Omega\overline{\varphi(x, k)}\,f(x)dx,$$

(6.6) $$\hat{g}(k)=(2\pi)^{-\frac{3}{2}}\int_\Omega\overline{\varphi(x, k)}\,g(x)dx.$$

Letting $\alpha\to\nu_{n-1}$, $\beta\to\nu_n$, and then summing up with respect to n, we obtain

(6.7) $$(f, g)=\int_M\hat{f}(k)\,\overline{\hat{g}(k)}\,dk,$$

by virtue of the property of the spectrum of H. So far Z was only defined for $f(x)\in C_0(\Omega)$. But the extension can be made in the obvious way.

To define Z', let us consider the integral

(6.8) $$L_g(f)=\int_M g(k)\,\overline{\hat{f}(k)}\,dk=(2\pi)^{-\frac{3}{2}}\int_M g(k)\left\{\int_\Omega\varphi(x, k)\,\overline{f(x)}dx\right\}dk,$$

where $f(x)\in C_0(\Omega)$, $g(k)\in C_0(M)$. Moreover the carrier of $g(k)$ is supposed to be disjoint with the set $\{k; |k|=\nu_n$ for some $n\}$. This condition for $g(k)$ enables the calculation somewhat easier. $L_g(f)$ defines a bounded conjugate linear functional on $L_2(\Omega)$. Therefore by means of Riesz's theorem, there exists a unique $g^*(x)$ in $L_2(\Omega)$ such that

(6.9) $$L_g(f)=(g^*, f)$$

holds. The correspondence between $g(x)$ and $g^*(x)$ is nothing but the transformation Z', of which norm apparently does not exceed 1.

The proof of (iii) of the theorem seems rather straightforward and is omitted here. See [1]. The formula

$$(E_\lambda f, g) = \int_{|k|<\sqrt{\lambda}} \hat{f}(k)\overline{\hat{g}(k)}dk \tag{6.10}$$

immediately leads to the proof of (v) of the theorem.

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