144. A Note on the Functional-Representations of Normal Operators in Hilbert Spaces

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Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let both $\{\varphi_\nu\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\cdots}$ be incomplete orthonormal infinite sets which are orthogonal to each other and by which a complete orthonormal system in $\mathfrak{H}$ is constructed; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrarily prescribed bounded sequence of complex numbers; let (u_{ij}) be an infinite unitary matrix with $|u_{jj}|<1, j=1, 2, 3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$; let N be the operator defined by

$$Nx=\sum_{\nu=1}^{\infty}\lambda_\nu(x, \varphi_\nu)\varphi_\nu+c\sum_{\mu=1}^{\infty}(x, \psi_\mu)\Psi_\mu$$

for every $x\in\mathfrak{H}$ and an arbitrarily given complex constant c; let L_y be the continuous linear functional associated with an arbitrary element $y\in\mathfrak{H}$; and let the operator N defined above be denoted symbolically by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}.$$

Then Nx is expressible in the form

$$Nx=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}(x) \quad (x\in\mathfrak{H}).$$

In Proceedings of the Japan Academy, Vol. 37, 614–618 (1961), I defined "the functional-representation of N" by $\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}$ and proved that the functional-representation of N converges uniformly, that N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$, and that $||N||=\max(\sup_\nu|\lambda_\nu|, |c|)$. In the same Proceedings, Vol. 38, 18–22 (1962), conversely I treated of the question as to whether any bounded normal operator with point spectrum in $\mathfrak{H}$ can always be expressed in the form of the above-mentioned infinite series of the continuous linear functionals associated with all the elements of a complete orthonormal system in $\mathfrak{H}$, by using such a unitary matrix as above. Though, in the latter paper, the conclusion was affirmative, an additional hypothesis, that is, "If the whole subset with non-zero measure of the continuous spectrum of N lies on a circumference with center at the origin" had to be set up: for otherwise, in the particular case where N has no eigenvalue, N is not necessarily expressed by the linear combination of L_{ψ_μ} in connection with the unitary matrix (u_{ij}), as Mr. D. A. Edwards pointed out in Mathematical Reviews, Vol. 26, No. 2 (1963).

In the present paper we shall show that the above-mentioned functional-representation replaced by a bounded Hermite matrix (α_{ij}) instead of the unitary matrix (u_{ij}) also expresses a bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $\mathfrak{H}$.

Theorem A. Let $\{\varphi_\nu\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\cdots}$ both be incomplete orthonormal infinite sets which are orthogonal to each other and by which a complete orthonormal system in $\mathfrak{H}$ is constructed; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrarily prescribed bounded sequence of complex numbers; let (α_{ij}) be an infinite Hermite matrix with $\overline{\alpha}_{ij}=\alpha_{ji}$ and $\sum_{k=1}^{\infty}|\alpha_{jk}|^2 \neq |\alpha_{jj}|^2$ such that the operator A associated with (α_{ij}) is a bounded operator in Hilbert coordinate space l_2; let $\Psi_\mu=\sum_{j=1}^{\infty}\alpha_{\mu j}\psi_j$; let L_x be the continuous linear functional associated with $x\in\mathfrak{H}$, that is, let $L_x(y)=(y, x)$ for every $y\in\mathfrak{H}$; and let N be the operator defined by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu},$$

where c is an arbitrarily given complex constant. Then this functional-representation of N converges uniformly and N is a bounded normal operator with point spectrum $\{\lambda_\nu\}_{\nu=1,2,3,\ldots}$, the norm of which is given by $\max(\sup_\nu|\lambda_\nu|, |c|\cdot||A||)$.

Proof. Since, by hypotheses, a complete orthonormal system in $\mathfrak{H}$ is constructed by the two incomplete orthonormal sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$, every element $x\in\mathfrak{H}$ is expressed in the form

$$x=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b_\mu\psi_\mu$$

where $a_\nu=L_{\varphi_\nu}(x)$ and $b_\mu=L_{\psi_\mu}(x)$, and $||x||^2=\sum_{\nu=1}^{\infty}|a_\nu|^2+\sum_{\mu=1}^{\infty}|b_\mu|^2<\infty$. Since, in addition, $\sum_{i=1}^{\infty}|\overline{\alpha}_{ij}|^2=\sum_{i=1}^{\infty}|\alpha_{ji}|^2<\infty$, $j=1, 2, 3,\cdots$, by virtue of the hypothesis concerning A, there is no difficulty in showing that

$$\begin{aligned}||Nx||^2&=||\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}(x)||^2 \quad (x\in\mathfrak{H})\\&=\sum_{\nu=1}^{\infty}|\lambda_\nu|^2|a_\nu|^2+|c|^2\sum_{k=1}^{\infty}|\sum_{j=1}^{\infty}b_j\alpha_{jk}|^2,\end{aligned}$$

and that

$$\begin{aligned}||Af||^2&=\sum_{k=1}^{\infty}|\sum_{j=1}^{\infty}b_j\alpha_{jk}|^2 \quad (f\equiv(\overline{b}_1, \overline{b}_2, \overline{b}_3,\cdots)\in l_2)\\&\leqq||A||^2||f||^2<\infty.\end{aligned}$$

Accordingly

$$||Nx||^2\leqq\sum_{\nu=1}^{\infty}|\lambda_\nu|^2|a_\nu|^2+|c|^2||A||^2\sum_{\mu=1}^{\infty}|b_\mu|^2\leqq M^2||x||^2,$$

where $M=\max(\sup_\nu|\lambda_\nu|, |c|\cdot||A||)$. Moreover, if x is an element belonging to the subspace determined by φ_ν, $||Nx||=|\lambda_\nu|\,||x||$; and if, on the contrary, x is in the subspace determined by $\{\psi_\mu\}$,

$$||Nx||=|c|\,||A\tilde{x}||\leqq|c|\,||A||\,||\tilde{x}||=|c|\,||A||\,||x||$$

where $\tilde{x}=(\overline{L_{\psi_1}(x)}, \overline{L_{\psi_2}(x)}, \overline{L_{\psi_3}(x)},\cdots)\in l_2$. In consequence, N is a bounded operator with norm M in $\mathfrak{H}$.

If we now denote by f_P the element derived from the above-mentioned element $f=(\bar{b}_1, \bar{b}_2, \bar{b}_3,\cdots)\in l_2$ by putting $\bar{b}_1=\bar{b}_2=\bar{b}_3=\cdots=\bar{b}_{P-1}=0$, then similarly it is verified without difficulty that, for any $x=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b_\mu\psi_\mu\in\mathfrak{H}$ where $a_\nu=L_{\varphi_\nu}(x)$ and $b_\mu=L_{\psi_\mu}(x)$,

$$\begin{aligned}||\sum_{\nu=P}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=P}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}(x)||^2&=\sum_{\nu=P}^{\infty}|\lambda_\nu|^2|a_\nu|^2+|c|^2\sum_{k=1}^{\infty}|\sum_{j=P}^{\infty}b_j\alpha_{jk}|^2\\&=\sum_{\nu=P}^{\infty}|\lambda_\nu|^2|a_\nu|^2+|c|^2||Af_P||^2\\&\leqq M^2(\sum_{\nu=P}^{\infty}|a_\nu|^2+\sum_{\mu=P}^{\infty}|b_\mu|^2).\end{aligned}$$

The positive integer P here can be so chosen as to satisfy the inequality

$$\sum_{\nu=P}^{\infty}|a_\nu|^2+\sum_{\mu=P}^{\infty}|b_\mu|^2<\frac{\varepsilon||x||^2}{M^2}$$

for an arbitrarily given positive number ε and any non-null element $x\in\mathfrak{H}$. Hence we have

$$||\sum_{\nu=P}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=P}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}||<\sqrt{\varepsilon}$$

for such a P. Thus the functional-representation of N converges uniformly.

Next we shall show that the operator N is normal. Since the identity operator I is given by $I=\sum_{\nu=1}^{\infty}\varphi_\nu\otimes L_{\varphi_\nu}+\sum_{\mu=1}^{\infty}\psi_\mu\otimes L_{\psi_\mu}$, it is found by direct computation that

$$\begin{aligned}(Nx, y)&=(\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}[\sum_{j=1}^{\infty}\alpha_{\mu j}\psi_j]\otimes L_{\psi_\mu}(x), \sum_{\nu=1}^{\infty}\varphi_\nu\otimes L_{\varphi_\nu}(y)\\&\qquad+\sum_{\mu=1}^{\infty}\psi_\mu\otimes L_{\psi_\mu}(y))\\&=\sum_{\nu=1}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\overline{L_{\varphi_\nu}(y)}+c\sum_{\kappa=1}^{\infty}\sum_{\mu=1}^{\infty}\alpha_{\mu\kappa}L_{\psi_\mu}(x)\overline{L_{\psi_\kappa}(y)}\quad(x, y\in\mathfrak{H}).\end{aligned}$$

Putting $\Psi_\mu^*=\sum_{j=1}^{\infty}\bar{\alpha}_{j\mu}\psi_j$ and $\bar{N}=\sum_{\nu=1}^{\infty}\bar{\lambda}_\nu\varphi_\nu\otimes L_{\varphi_\nu}+\bar{c}\sum_{\mu=1}^{\infty}\Psi_\mu^*\otimes L_{\psi_\mu}$, similarly we can show that the functional-representation of $\bar{N}$ is uniformly convergent, that $\bar{N}$ is a bounded operator in $\mathfrak{H}$, and that

$$(x, \bar{N}y)=\sum_{\nu=1}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\overline{L_{\varphi_\nu}(y)}+c\sum_{\kappa=1}^{\infty}\sum_{\mu=1}^{\infty}\alpha_{\mu\kappa}L_{\psi_\mu}(x)\overline{L_{\psi_\kappa}(y)}\quad(x, y\in\mathfrak{H}).$$

We have therefore $(Nx, y)=(x, \bar{N}y)$, which implies that $\bar{N}$ is the adjoint operator N^* of N. In addition, it is a matter of simple manipulations to show that

$$NN^*x = N[\sum_{\nu=1}^{\infty}\bar{\lambda}_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+\bar{c}\sum_{\mu=1}^{\infty}\Psi_\mu^*\otimes L_{\phi_\mu}(x)] \quad (x\in\mathfrak{H})$$
$$= \sum_{\nu=1}^{\infty}|\lambda_\nu|^2 L_{\varphi_\nu}(x)\varphi_\nu + |c|^2\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\bar{\alpha}_{\mu\kappa}L_{\phi_\kappa}(x)]\Psi_\mu$$

and that

$$N^*Nx = N^*[\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\phi_\mu}(x)] \quad (x\in\mathfrak{H})$$
$$= \sum_{\nu=1}^{\infty}|\lambda_\nu|^2 L_{\varphi_\nu}(x)\varphi_\nu + |c|^2\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\alpha_{\kappa\mu}L_{\phi_\kappa}(x)]\Psi_\mu^*.$$

Since, on the other hand, $\bar{\alpha}_{\mu\kappa}=\alpha_{\kappa\mu}$ for $\mu, \kappa=1, 2, 3, \cdots$ by the hypothesis on the matrix (α_{ij}), and hence since $\Psi_\mu^*=\Psi_\mu$, the just established results permit us to conclude that $NN^*=N^*N$ in $\mathfrak{H}$. Consequently N is a normal operator in $\mathfrak{H}$.

Thus it remains only to prove that the set $\{\lambda_\nu\}$ is the point spectrum of N. However it is obvious that any λ_ν is an eigenvalue of N corresponding to the eigenelement φ_ν; and moreover, since $\sum_{\kappa=1}^{\infty}|\alpha_j x|^2 \neq |\alpha_{jj}|^2$, N has not any eigenvalue other than λ_ν, $\nu=1, 2, 3, \cdots$, as can be seen from the reasoning used in one of the preceding papers [cf. Proc. Japan Acad., Vol. 37, 614–618 (1961)]. Consequently the point spectrum of N is given by $\{\lambda_\nu\}$ itself.

Remark A. Though this theorem holds also in the case where $\{\lambda_\nu\}$ is a finite set, we are interested in the case where $\{\lambda_\nu\}$ is an infinite set. Because, by applying the bounded normal operator defined by an arbitrarily given functional-representation $\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu} + c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\phi_\mu}$ where Ψ_μ denotes such an element $\sum_{j=1}^{\infty}u_{\mu j}\psi_j$ or $\sum_{j=1}^{\infty}\alpha_{\mu j}\psi_j\in\mathfrak{H}$ as was described before, we can treat of various problems on complex-valued functions which cannot be discussed from a point of view of the classical function theory.

Remark B. Let N be the bounded normal operator defined by such a functional-representation as was stated in Remark A; let Δ_a be the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself; let Δ be the continuous spectrum of N; let $\Delta'=\Delta-\Delta_a$; and let $\{K(\lambda)\}$ be the spectral family of N. Since the projector $K(\Delta')$ is permutable with each of N and N^*,

$$N(I-K(\Delta'))\cdot[N(I-K(\Delta'))]^* = [N(I-K(\Delta'))]^*\cdot N(I-K(\Delta'))$$

in $\mathfrak{H}$. $N(I-K(\Delta'))$ is therefore a bounded normal operator. Furthermore it is readily verified that not only $\{\lambda_\nu\}$ is the point spectrum of $N(I-K(\Delta'))$, but that also

$$N(I-K(\Delta')) = \int_{\{\lambda_\nu\}\cup\Delta_a}\lambda dK(\lambda);$$

and hence it is found that Δ_a is the continuous spectrum of $N(I-K(\Delta'))$. This result is useful for applications of the spectral theory to the function theory.