139. A Note on Intra-regular Semigroups

By Sandor LAJOS
University of Economics, Budapest, Hungary
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Following Clifford and Preston [2] we shall say that a semigroup S is intra-regular if, for any element a of S, there exist x and y in S such that

$$xa^2y=a,$$

that is, S is intra-regular if $a\in Sa^2S$, for all a in S.

It is easy to see that e.g. every semigroup admitting relative inverses (see: [1]) is an intra-regular semigroup. A subset X of a semigroup S is called semiprime if $a^2\in X$, $a\in S$ imply $a\in X$. It is known that a semigroup S is intra-regular if and only if every two-sided ideal of S is semiprime (see: [2] Lemma 4.1). Throughout this paper, by ideal we shall mean two-sided ideal. For the fundamental concepts of the algebraic theory of semigroups we refer to [2] and [4].

In this note we prove the following

Theorem. *Every ideal of an ideal of an intra-regular semigroup S is an ideal of S.*

For the proof we need the following

Lemma 1. *In an intra-regular semigroup every ideal A is idempotent, that is, $A^2=A$.*

Proof. Let S be an intra-regular semigroup, and let A be an ideal of s. We prove that $A\subseteq A^2$. Let $a\in A$, then $a^2\in A^2$. This implies that $a\in A^2$, because A^2 is an ideal of S, and every ideal of S is semiprime. Since $A^2\subseteq A$, we obtain that $A^2=A$. Thus every ideal of S is idempotent.

Lemma 2. *Let S be an arbitrary semigroup, and let B be an ideal of an ideal a of S. Denote by $\overline{B}$ the ideal of S generated by B. Then $\overline{B}^3\subseteq B$.*

Proof. Since

$$\overline{B}=B\cup BS\cup SB\cup SBS,$$

it follows that

$$\overline{B}^3\subseteq A\overline{B}A=A(B\cup BS\cup SB\cup SBS)A\subseteq B,$$

which we wished to prove.

Now we can prove the above theorem. Let S be an intra-regular semigroup. Let A be an ideal of S, and let B be an ideal of the semigroup A. Denote by $\overline{B}$ the ideal of S generated by B. Then by Lemma 1, $\overline{B}^2=\overline{B}$. This and Lemma 2 imply $\overline{B}\subseteq B$. Since $B\subseteq\overline{B}$ evidently holds, we have $\overline{B}=B$. Therefore B is an ideal of S, that is, our theorem is proved.

It naturally raises the following

Problem. Characterize the class of all semigroups S which have the property that every ideal of an ideal is an ideal of S.

Note that the semigroups in which every ideal is idempotent (thus for example the regular semigroups) have this property (see: [3]), and easy to show that β-semigroups (see: [5]) also have this property.

References

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