137. On Algebraic Varieties Uniformizable by Bounded Domains

By Tetsuji SHIODA
Department of Mathematics, University of Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., Nov. 12, 1963)

Let D be a bounded domain in C^n and Γ a discontinuous group of analytic automorphisms of D. We suppose throughout this paper (unless otherwise mentioned) that Γ satisfies the following two conditions:

(1) No element of Γ except the identity has a fixed point in D.

(2) $V=\Gamma\backslash D$ is compact.

Then V is, as is well-known, analytically equivalent to a non-singular projective algebraic variety. (Cf. [2], [5] Theorem 3 or [6] Theorem 6.) We consider in the following algebraic varieties V, expressible in the form $\Gamma\backslash D$, where D and Γ are as described above.

$\mathfrak{A}(V)$ will denote the group of all analytic automorphisms of V, $K(V)$ the field of all meromorphic functions on V, and $\mathfrak{A}(K(V))$ the group of all automorphisms of $K(V)$ over C.

Igusa [5] proved, in case D is a hypersphere in C^n:

$$|z_1|^2+\cdots+|z_n|^2<1,$$

that (a) V is a minimal model, and (b) $\mathfrak{A}(K(V))$ is a finite group.

We shall show that (i) (a) holds in general (for any bounded domain D), (ii) $\mathfrak{A}(V)$ is finite, if D is simply-connected, and (iii) $\mathfrak{A}(V)\cong\mathfrak{A}(K(V))$. (ii), (iii) imply of course (b). To prove (i) and (iii), we can utilize the idea of [5] (Theorems 6 and 7, p. 675), and the result (ii) can be found in [3], [7]. However we shall also give a proof of (ii) for completeness' sake.

It is well-known that any compact Riemann surface V of genus ≥ 2 can be uniformized by $D=$ the unit disc of the complex plane. In this case, the fact that $\mathfrak{A}(V)$ is finite is a classical Schwarz-Klein theorem, and it is also known that $\mathfrak{A}(V)\cong\mathfrak{A}(K(V))$. Thus our results contain generalizations of these classical facts.

The author wishes to express his sincere gratitude to Professors S. Iyanaga and J. Igusa for their valuable advices.

§1. A complex manifold M imbedded in a projective space will be called "minimal" if any meromorphic mapping from any complex manifold U into M is necessarily holomorphic. Here a meromorphic mapping is defined invariantly using the inhomogeneous coordinates of M and any local coordinates of U. (Cf. [5], p. 674.)

Theorem 1. *Notations being as above, the variety* $V=\Gamma\backslash D$ *is* *minimal.*

Proof. Let F be a meromorphic mapping from U^m (a complex manifold of dimension m) into V. We shall show that F is holomorphic. We may assume that U is a connected and simply-connected neighborhood of the origin of $\boldsymbol{C}^m$. The "fundamental locus" W of F (i.e. the set of points of U, at which F is not holomorphic,) is an analytic set of dimension $\leq m-2$, since V is imbedded in a projective space $\boldsymbol{P}^N$ and W can be considered as common zeros of the functions on U representing the homogeneous coordinates of V. Therefore $U-W$ is connected and simply-connected. The restriction of F to $U-W$ is a holomorphic mapping from $U-W$ into V.

Let z_0 be a point of $U-W$ and let P_0 be a point of the covering manifold (D, π) of V lying over $F(z_0)$. (π means the natural projection $D\rightarrow V=\Gamma\backslash D$.) We show that we can define a (single-valued) holomorphic mapping F^* from $U-W$ into D such that $F^*(z_0)=P_0$ and $\pi\circ F^*=F$.

For any point z_1 of $U-W$, we connect z_0 and z_1 by a continuous curve z_t $(0\leq t\leq 1)$ in $U-W$. Over the curve $F(z_t)$ $(0\leq t\leq 1)$ in V, there lies a unique curve P_t $(0\leq t\leq 1)$ in D starting from P_0. We define F^* by $F^*(z_1)=P_1$. Since $U-W$ is simply-connected, this definition is independent of curves z_t connecting z_0 and z_1, and F^* gives a single-valued mapping with desired properties.

Now any component F^*_α of F^* is a *bounded* holomorphic function on $U-W$ $(\alpha=1,\cdots,n)$. By a theorem of Hartogs F^*_α must be holomorphic throughout U. Consequently $F=\pi\circ F^*$ is holomorphic throughout U. Q.E.D.

Remark. Also when the condition (2) for Γ is not satisfied, $V=\Gamma\backslash D$ is a minimal complex manifold, if V has a projective-embeddable compactification $\overline{V}$, as is clear from the above proof. But we do not know whether $\overline{V}$ itself is minimal or not, when V is not compact. In fact we know nothing about the universal covering of $\overline{V}$, even when $\overline{V}$ is non-singular.

§2. **Theorem 2.** *If D is simply-connected, then $\mathfrak{A}(V)$ is a finite group.*

Proof. First every analytic automorphism of V is induced by an analytic automorphism of D, since D is the universal covering of V. Conversely an analytic automorphism of D induces that of V if and only if it belongs to the normalizer $N(\Gamma)$ of Γ in $\mathfrak{A}(D)$. Hence

$$(*)\qquad \mathfrak{A}(V)\cong N(\Gamma)/\Gamma,$$

the isomorphic mapping being given in the natural way. This isomorphism gives also an isomorphism between topological groups $\mathfrak{A}(V)$ and $N(\Gamma)/\Gamma$, the compact-uniform topology being introduced into

$\mathfrak{A}(V)$ and $N(\Gamma)$, and the quotient topology into $N(\Gamma)/\Gamma$. By a theorem of Bochner-Montgomery, the topological group $\mathfrak{A}(V)$ becomes a complex Lie group. ([1], Theorem 1.) Since Γ is discrete in $N(\Gamma)$, we can consider $N(\Gamma)$ as complex Lie group by the isomorphism $(*)$. Thus the complex Lie group $N(\Gamma)$ acts on the bounded domain D, and so $N(\Gamma)$ is discrete. (Cf. [1], Theorem 2.) Consequently $N(\Gamma)$ is a discontinuous group, and

$$\#\mathfrak{A}(V)=[N(\Gamma):\ \Gamma]=v(\Gamma)/v(N(\Gamma))<+\infty$$

where $v(\Gamma)$ denotes the invariant volume of the fundamental domain for Γ in D, and $v(N(\Gamma))$ has the similar meaning for $N(\Gamma)$. Q.E.D.

Remark. When D is a symmetric bounded domain, and $N(\Gamma)$ satisfies the condition (1) for Γ, then we have

$$\#\mathfrak{A}(V)=\chi(\Gamma\backslash D):\ \chi(N(\Gamma)\backslash D),$$

where $\chi(V)$ denotes the Euler-Poincaré characteristic of V. This follows from the above proof and the "proportionality relation" of Hirzebruch. ([4], Satz 3 and 4.)

§3. **Theorem 3.** *$\mathfrak{A}(K(V))$ is isomorphic to $\mathfrak{A}(V)$.*

Proof. For an arbitrary element σ of $\mathfrak{A}(V)$, we define $\bar{\sigma}\in\mathfrak{A}(K(V))$ by $\bar{\sigma}(f)=f\circ\sigma^{-1}$, f being any function in $K(V)$. $\sigma\to\bar{\sigma}$ is clearly an injective homomorphism. We show that this is surjective.

Let φ be any element of $\mathfrak{A}(K(V))$, and let $(x_0, x_1,\cdots, x_N)$ be a generic point of $V\subset \boldsymbol{P}^N$. Then $K(V)=\boldsymbol{C}(x_1/x_0,\cdots, x_N/x_0)$. Define σ by

$$(1, x_1/x_0,\cdots, x_N/x_0)\to(1, \varphi(x_1/x_0),\cdots, \varphi(x_N/x_0)).$$

Then σ is a birational mapping from V to itself. Since V is minimal by Theorem 1, a birational mapping is necessarily biregular, i.e. $\sigma\in\mathfrak{A}(V)$, and it is easy to see $\overline{\sigma^{-1}}=\varphi$. Q.E.D.

Remark. This proof is based only on the minimality of V.

Corollary. *If D is simply-connected, $\mathfrak{A}(K(V))$ is a finite group.*

References

[1] S. Bochner and D. Montogomery: Groups on analytic manifolds, Ann. of Math., **48**, 659–669 (1947).

[2] H. Cartan: Quotient d'un espace analytique par un groupe d'automorphismes, Algebraic geometry and topology. A symposium in honor of Lefschetz, Princeton, 90–102 (1957).

[3] N. S. Hawley: A theorem on compact complex manifolds, Ann. of Math., **52**, 637–641 (1950).

[4] F. Hirzebruch: Automorphen Formen und der Satz von Riemann-Roch, Symposium internacional de topologia algebraica, Mexico, 129–144 (1958).

[5] J. Igusa: On the structure of a certain class of Kaeler varieties, Amer. J. Math., **76**, 669–678 (1954).

[6] K. Kodaira: On Kähler varieties of restricted type, Ann. of Math., **60**, 28–48 (1954).

[7] J. H. Sampson: A note on automorphic varieties, Proc. Nat. Acad. Sci. U.S.A., **38**, 895–898 (1952).

[8] A. Weil: Introduction à l'étude des variétés kählériennes, Paris (1957).