165. On the Uniqueness of Balayaged Measures

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Introduction. Let Ω be a locally compact Hausdorff space, every compact subset of which is separable, and $G(x, y)$ be a positive continuous (in the extended sense) kernel on Ω.[1] In [2], we proved that a regular symmetric balayable kernel G satisfies the U- and BU-principles[2] if and only if it is non-degenerate, that is, for any different points x_1 and x_2 in Ω,

$$G(x, x_1)/G(x, x_2) \neq \text{any constant in } \Omega.$$

In this paper we shall extend this result to non-symmetric kernels.

§1. Non-degeneracy. Theorem 1. *If G satisfies the U- or BU-principle, it is non-degenerate.*

This is evident.

Theorem 2. *Let G be non-degenerate and satisfy*

(i) *the domination principle or*

(ii) *the balayage principle and the continuity principle. Then its adjoint kernel $\check{G}$ is non-degenerate.*

Proof. If G satisfies the condition (ii), then it satisfies (i).[3] Therefore we may assume that G satisfies the domination principle. Contrary suppose that $\check{G}$ is degenerate. Then there are different points x_1 and x_2 such that $G(x_1, x) = aG(x_2, x)$ for any point x in Ω with a positive constant a. Then $G(x_1, x_1)$ and $G(x_2, x_2)$ are finite and $G\varepsilon_{x_1}(x_i) = bG\varepsilon_{x_2}(x_i)$ $(i=1, 2)$ with $b = G(x_1, x_1)/G(x_1, x_2)$. Hence by the domination principle

$$G\varepsilon_{x_1} = bG\varepsilon_{x_2} \quad \text{in } \Omega.$$

This shows that G is degenerate.

Corollary. *The adjoint kernel $\check{G}$ is non-degenerate if and only if G is non-degenerate, provided that*

(i) *G satisfies the domination principle and $\check{G}$ satisfies the continuity principle or*

(ii) *G satisfies the balayage principle and the continuity principle.*

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1) We use the same notations as in [3].

2) The U-principle means that if G-potentials of positive measures with compact support coincide with each other G-p.p.p. in Ω, then the measures are identical.

The BU-principle means that the G-balayaged measure is uniquely determined by a given positive measure and a compact set.

3) Cf. [3, 4].

§2. **U-principle.** Let K be a compact subset of Ω and $\mathfrak{C}(K)$ be the space of all finite continuous functions on K with the uniform convergence topology. We denote by $\mathfrak{D}(K)$ the subspace of $\mathfrak{C}(K)$ consisting of functions f which are $\check{G}$-potentials of signed measures, that is, $f=\check{G}\mu_1-\check{G}\mu_2$ with $\mu_1, \mu_2 \in \mathfrak{M}_0$.[4]

Theorem 3. *Let G satisfy the balayage principle and the continuity principle. If G is non-degenerate and K is $\check{G}$-regular,*[5] *then $\mathfrak{D}(K)$ is dense in $\mathfrak{C}(K)$.*

Proof. This follows from the following two remarks.

(a) $\mathfrak{D}(K)$ is closed with respect to the operations $\vee$ and $\wedge$.[6] In fact, let $\check{G}\mu_1$ and $\check{G}\mu_2$ be $\check{G}$-potentials in $\mathfrak{D}(K)$ with $\mu_i \in \mathfrak{M}_0$ $(i=1, 2)$ and put $u=\check{G}\mu_1 \wedge \check{G}\mu_2$. By the existence theorem[7] there exists a positive measure μ, supported by K, such that

$$\check{G}\mu \geq u \quad G\text{-p.p.p. on } K,$$
$$\check{G}\mu \leq u \quad \text{on } S\mu.$$

Since G satisfies the balayage principle and the continuity principle, $\check{G}$ satisfies the domination principle.[8] Hence by the above inequalities we obtain

$$\check{G}\mu \leq u \quad \text{in } \Omega,$$
$$\check{G}\mu = u \quad G\text{-p.p.p. on } K.$$

Moreover by the regularity of K we have

$$\check{G}\mu \geq u \quad \text{on } K.$$

Consequently $\check{G}\mu=u$ on K. This shows that $\check{G}\mu_1 \wedge \check{G}\mu_2$ belongs to $\mathfrak{D}(K)$. From this it follows immediately that $\mathfrak{D}(K)$ is closed with respect to $\vee$ and $\wedge$.

(b) For any different two points x_1, x_2 on K and any real numbers a_1, a_2, there exists a function f in $\mathfrak{D}(K)$ such that $f(x_i)=a_i$ $(i=1, 2)$. In fact, G being non-degenerate, there exist different two points y_1 and y_2 on K such that

$$y_i \neq x_1, x_2 \quad (i=1, 2),$$
$$\frac{G(y_1, x_1)}{G(y_1, x_2)} \neq \frac{G(y_2, x_1)}{G(y_2, x_2)}.$$

We can take a positive measure λ such that $\check{G}\lambda$ belongs to $\mathfrak{D}(K)$ and $\check{G}\lambda(x_i) > \check{G}\varepsilon_{y_1}(x_i)$ $(i=1, 2)$. Then there exists a positive measure μ_1,

4) $\mathfrak{M}_0$ is the totality of positive measures with compact support.

5) Namely an inequality $\check{G}\mu \geq h$ G-p.p.p. on K for $\mu \in \mathfrak{M}_0$ and a positive finite continuous function h on K implies $\check{G}\mu \geq h$ everywhere on K.

6) $(f \vee g)(x)=\max\{f(x), g(x)\}$, $(f \wedge g)(x)=\min\{f(x), g(x)\}$.

7) Cf. [3, 4].

8) Cf. [3, 4].

supported by K, such that $\check{G}\mu_1 \in \mathfrak{D}(K)$, $\check{G}\mu_1 = \check{G}\lambda \wedge \check{G}\varepsilon_{y_1}$ on K, and hence $\check{G}\mu_1(x_i) = \check{G}\varepsilon_{y_1}(x_i)$. Similarly we have a positive measure μ_2, supported by K, such that $\check{G}\mu_2 \in \mathfrak{D}(K)$ and $\check{G}\mu_2(x_i) = \check{G}\varepsilon_{y_2}(x_i)$ $(i=1, 2)$.

Now we take real numbers t_1 and t_2 such that

$$t_1\check{G}\mu_1(x_1) + t_2\check{G}\mu_2(x_1) = a_1$$
$$t_1\check{G}\mu_1(x_2) + t_2\check{G}\mu_2(x_2) = a_2.$$

Then $f = t_1\check{G}\mu_1 + t_2\check{G}\mu_2$ belongs to $\mathfrak{D}(K)$ and $f(x_i) = a_i$. This proves (b).

Our theorem follows from (a) and (b) by the theorem of Weierstrass-Stone.[9]

Theorem 4. *Let G satisfy the balayage principle and the continuity principle, and $\check{G}$ be regular.*[10] *If G is non-degenerate, then G satisfies the U-principle.*

Proof. Let $G\nu_1 = G\nu_2$ G-p.p.p. in Ω with $\nu_1, \nu_2 \in \mathfrak{M}_0$. We take a $\check{G}$-regular compact set K containing $S\nu_1 \cup S\nu_2$. Then by the preceding theorem $\mathfrak{D}(K)$ is dense in $\mathfrak{C}(K)$. From this follows immediately our theorem.

§3. Uniqueness of balayaged measures. Lemma. *Let G satisfy the balayage principle and the continuity principle, and let $\check{G}$ be regular. Then G-balayaged potentials are uniquely determined.*

Proof. Let $G\nu_i$ $(i=1,2)$ be G-balayaged potentials of ν on K. Then $G\nu_1 = G\nu_2$ G-p.p.p. on K. Take a point x_0 in $\Omega - K$. As $\check{G}$ is regular, there exists a $\check{G}$-regular compact set $K' \supset K$ which does not contain x_0. Then $\check{G}$-balayaged potential $\check{G}\varepsilon'$ of ε_{x_0} on K' coincides everywhere on K' with $\check{G}\varepsilon_{x_0}$.[11] Consequently

$$G\nu_1(x_0) = \int \check{G}\varepsilon_{x_0}\, d\nu_1 = \int \check{G}\varepsilon'\, d\nu_1 = \int G\nu_1\, d\varepsilon' = \int G\nu_2\, d\varepsilon' = G\nu_2(x_0).$$

This proves that $G\nu_1 = G\nu_2$ everywhere in $\Omega - K$. Consequently $G\nu_1 = G\nu_2$ G-p.p.p. in Ω.

Theorem 5. *Let G satisfy the balayage principle and the continuity principle, and let $\check{G}$ be regular. If G is non-degenerate, then it satisfies the BU-principle.*

Proof. This is an immediate consequence of Theorem 3 and the above lemma.

Summarizing up the preceding results we have

Theorem 6. *Assume that G satisfies the balayage principle and*

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9) Cf. [1].

10) Namely for any compact set K and its open neighborhood ω, there exists a $\check{G}$-regular compact set K' with $K \subset K' \subset \omega$.

11) G satisfies the balayage principle (cf. [3, 4]).

the continuity principle and that the adjoint kernel $\check{G}$ is regular. Then, in order that G satisfies the U- and BU-principles, it is necessary and sufficient that G is non-degenerate.

§4. U-principle with respect to the adjoint kernel. Now we consider whether $\check{G}$ satisfies the U-principle provided that G satisfies it. The answer to this problem is negative in general. In fact, let Ω be an open interval $\{|x|<1\}$ in the 1-dimensional Euclidean space and let $G(x, y)$ be given by

$$G(x, y)=\sum_{n=0}^{\infty} x^{2n}y^{n}.$$

Then G satisfies the U-principle but $\check{G}$ does not.

If G satisfies the balayage principle and other additional conditions, the answer is affirmative.

Theorem 7. *Assume that G is regular and satisfies the balayage principle and the continuity principle, and that the adjoint kernel $\check{G}$ is regular.*[12] *Then $\check{G}$ satisfies the U-principle if and only if G does it.*

References

[1] N. Bourbaki: Topologie générale, Chap. X, Paris (1949).
[2] M. Kishi: Unicity principles in the potential theory, Osaka Math. J., **13**, 41–74 (1961).
[3] ——: Note on balayage and maximum principles, Proc. Japan Acad., **39**, 415–418 (1963).
[4] ——: Maximum principles in the potential theory, to appear.
[5] ——: Weak domination principle, to appear.

12) $\check{G}$ satisfies the continuity principle, since G satisfies the balayage principle (cf. [5]).