164. A Note on the Functional-Representations of Normal Operators in Hilbert Spaces. II

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In this paper we shall discuss the most general type of the functional-representations for normal operators in the abstract Hilbert space $\mathfrak{H}$ which is separable and infinite dimensional.

Lemma A. Let (β_{ij}) denote any infinite complex matrix

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \cdot & \cdot & \cdot \\ \beta_{21} & \beta_{22} & \beta_{23} & \cdot & \cdot & \cdot \\ \beta_{31} & \beta_{32} & \beta_{33} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$; and let B denote the operator associated with (β_{ij}) in Hilbert coordinate space l_2. Then, in order that the bounded operator B be normal in l_2, it is necessary and sufficient that $\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu}=\sum_{\nu=1}^{\infty}\bar{\beta}_{\nu i}\beta_{\nu j}$ for every pair of $i, j=1, 2, 3, \cdots$.

Proof. Since, by hypotheses, $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$, it is easily verified with the help of Cauchy's inequality that $\|B\tilde{x}\|^2\leqq\sum_{i,j=1}^{\infty}|\beta_{ij}|^2\cdot\|\tilde{x}\|^2$ for every $\tilde{x}\in l_2$. Hence B is a bounded operator in l_2. Now we consider the transposed matrix $(\bar{\beta}_{ij})^T$ of $(\bar{\beta}_{ij})$, which is obtained from $(\bar{\beta}_{ij})$ by interchanging rows and columns in $(\bar{\beta}_{ij})$, and denote by $\tilde{B}$ the operator associated with $(\bar{\beta}_{ij})^T$ in l_2. Then, for every pair of elements $\tilde{x}=(x_1, x_2, x_3,\cdots)$ and $\tilde{y}=(y_1, y_2, y_3,\cdots)$ belonging to l_2 we have

$$\begin{aligned}(\tilde{x}, \tilde{B}\tilde{y}) &= \sum_{j=1}^{\infty}[\sum_{i=1}^{\infty}\beta_{ij}\bar{y}_i]x_j \\ &= \sum_{i=1}^{\infty}[\sum_{j=1}^{\infty}\beta_{ij}x_j]\bar{y}_i \\ &= (B\tilde{x}, \tilde{y}),\end{aligned}$$

because the absolute convergency of these iterated infinite sums can be verified by virtue of the applications of Cauchy's inequality and the hypothesis $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$. Hence $\tilde{B}$ is the adjoint operator B^* of B in l_2. By making use of this result we can readily verify that BB^* is the bounded operator associated with the matrix $(\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu})$

where $\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu}$ denotes the element appearing in row i column j and that B^*B is the bounded operator associated with the matrix $(\sum_{\nu=1}^{\infty}\bar{\beta}_{\nu j}\beta_{\nu j})$ where the index i denotes the number of the row and the index j denotes the number of the column. In consequence, a necessary and sufficient condition that $BB^*=B^*B$ in l_2 is that $\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu} =\sum_{\nu=1}^{\infty}\bar{\beta}_{\nu i}\beta_{\nu j}$ for every pair of $i, j=1, 2, 3,\cdots$, as we were to prove.

Remark C. It is at once obvious that if (β_{ij}) is a unitary matrix or an Hermite matrix stated in the earlier discussion, the relation $\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu}=\sum_{\nu=1}^{\infty}\bar{\beta}_{\nu i}\beta_{\nu j}$ holds for every pair of $i, j=1, 2, 3,\cdots$. Besides these particular matrices, however, there are many matrices satisfying the just established relation. For example, the matrix $(\beta_{jk}) =\left(\frac{e^{i\theta}}{2^{(j+k)}}\right)$, $(i=\sqrt{-1}, 0<\theta<\pi)$, is a desired matrix which is neither unitary nor Hermitian.

Definition. Any infinite matrix (β_{ij}) satisfying the conditions $\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu}=\sum_{\nu=1}^{\infty}\bar{\beta}_{\nu i}\beta_{\nu j}$, $i, j=1, 2, 3,\cdots$, is called a normal matrix.

Theorem B. Let $\{\varphi_\nu\}_{\nu=1,2,3,\ldots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\ldots}$ both be incomplete orthonormal sets which are mutually orthogonal and by which a complete orthonormal system in the abstract Hilbert space $\mathfrak{H}$ is constructed; let $\{\lambda_\nu\}_{\nu=1,2,3,\ldots}$ be an arbitrarily prescribed bounded sequence of complex numbers (inclusive of the respective multiplicities); let L_x be the continuous linear functional associated with any $x\in\mathfrak{H}$; let (β_{ij}) be a bounded normal matrix with $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$ and $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2 \neq |\beta_{\mu\mu}|^2$, $\mu=1, 2, 3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}\beta_{\mu j}\psi_j$; let c be an arbitrarily given complex constant; and let N be the operator defined by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu} \tag{1}$$

in the sense of $Nx=\sum_{\nu=1}^{\infty}\lambda_\nu(x, \varphi_\nu)\varphi_\nu+c\sum_{\mu=1}^{\infty}(x, \psi_\mu)\Psi_\mu$, $(x\in\mathfrak{H})$. Then this functional-representation defining N converges uniformly and the N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $\mathfrak{H}$; and moreover $||N||=\max(\sup_\nu|\lambda_\nu|, |c|\cdot||B||)$ where B denotes the operator associated with the matrix (β_{ij}) in Hilbert coordinate space l_2.

Proof. From the hypothesis concerning (β_{ij}) it is found that the operator B associated with (β_{ij}) is a bounded operator in l_2, as already shown at the beginning of the proof of Lemma A. By the same methods as those used to prove Theorem A in the preceding paper, we can therefore show that

$$||Nx||^2=\sum_{\nu=1}^{\infty}|\lambda_\nu|^2|L_{\varphi_\nu}(x)|^2+|c|^2\sum_{\kappa=1}^{\infty}|\sum_{\mu=1}^{\infty}\beta_{\mu\kappa}L_{\psi_\mu}(x)|^2 \quad (x\in\mathfrak{H}),$$

and that

$$||B^*f||^2=\sum_{\kappa=1}^{\infty}|\sum_{\mu=1}^{\infty}\bar{\beta}_{\mu\kappa}\overline{L_{\phi_\mu}(x)}|^2 \quad (f=(\overline{L_{\phi_1}(x)}, \overline{L_{\phi_2}(x)}, \overline{L_{\phi_3}(x)},\cdots)\in l_2)$$
$$\leqq ||B^*||^2||f||^2=||B||^2||f||^2<\infty.$$

Accordingly

$$||Nx||^2\leqq\sum_{\nu=1}^{\infty}|\lambda_\nu|^2|L_{\varphi_\nu}(x)|^2+|c|^2||B||^2\sum_{\mu=1}^{\infty}|L_{\phi_\mu}(x)|^2 \quad (x\in\mathfrak{H})$$
$$\leqq M^2||x||^2 \quad (M=\max(\sup_\nu|\lambda_\nu|,\ |c|\cdot||B||)).$$

Moreover, if x is an element belonging to the subspace determined by a φ_ν, $||Nx||=|\lambda_\nu|\,||x||$; and if, on the contrary, x is in the subspace determined by the set $\{\psi_\mu\}$,

$$||Nx||=|c|\,||B^*f||\leqq|c|\,||B^*||\,||f||=|c|\,||B||\,||x|| \quad (f=(\overline{L_{\phi_1}(x)}, \overline{L_{\phi_2}(x)},\cdots)\in l_2).$$

Consequently N is a bounded operator with norm M in $\mathfrak{H}$.

Since, as can be found from the above discussion, it is easily verified that

$$||\sum_{\nu=p}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\varphi_\nu+c\sum_{\mu=p}^{\infty}L_{\phi_\mu}(x)\Psi_\mu||^2\leqq M^2(\sum_{\nu=p}^{\infty}|L_{\varphi_\nu}(x)|^2+\sum_{\mu=p}^{\infty}|L_{\phi_\mu}(x)|^2) \quad (x\in\mathfrak{H})$$

and hence that for an arbitrarily given positive number ε there exists a suitably large number G such that

$$||\sum_{\nu=p}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=p}^{\infty}\Psi_\mu\otimes L_{\phi_\mu}||<\varepsilon \quad (p\geqq G).$$

Hence the functional series on the right of (1) is uniformly convergent.

Next we consider the operator $\bar{N}$ defined by

$$\bar{N}=\sum_{\nu=1}^{\infty}\bar{\lambda}_\nu\varphi_\nu\otimes L_{\varphi_\nu}+\bar{c}\sum_{\mu=1}^{\infty}\Psi_\mu^*\otimes L_{\phi_\mu}$$

where $\Psi_\mu^*=\sum_{j=1}^{\infty}\bar{\beta}_{j\mu}\psi_j$, $\mu=1, 2, 3,\cdots$. Since, as in the proof of Theorem A, it is shown by direct computation that

$$(Nx, y)=\sum_{\nu=1}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\overline{L_{\varphi_\nu}(y)}+c\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\beta_{\kappa\mu}L_{\phi_\kappa}(x)]\overline{L_{\phi_\mu}(y)}=(x, \bar{N}y)$$

for every pair of $x, y\in\mathfrak{H}$, $\bar{N}$ is identical with the adjoint operator N^* of N. Hence

$$NN^*x=\sum_{\nu=1}^{\infty}|\lambda_\nu|^2L_{\varphi_\nu}(x)\varphi_\nu+|c|^2\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\bar{\beta}_{\mu\kappa}L_{\phi_\kappa}(x)]\Psi_\mu$$

and

$$N^*Nx=\sum_{\nu=1}^{\infty}|\lambda_\nu|^2L_{\varphi_\nu}(x)\varphi_\nu+|c|^2\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\beta_{\kappa\mu}L_{\phi_\kappa}(x)]\Psi_\mu^*$$

for every $x\in\mathfrak{H}$. On the other hand, since it is verified with the aid of the hypothesis $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$ and Cauchy's inequality that both $\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}|\bar{\beta}_{\mu\kappa}\beta_{\mu j}L_{\phi_\kappa}(x)|]$ and $\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}|\beta_{\kappa\mu}\bar{\beta}_{j\mu}L_{\phi_\kappa}(x)|]$ converge for $j=1, 2, 3, \cdots$, we have

$$\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\bar{\beta}_{\mu\kappa}L_{\phi_\kappa}(x)]\Psi_\mu=\sum_{j=1}^{\infty}[\sum_{\kappa=1}^{\infty}\sum_{\mu=1}^{\infty}\bar{\beta}_{\mu\kappa}\beta_{\mu j}L_{\phi_\kappa}(x)]\psi_j$$

and

$$\sum_{\mu=1}^{\infty}[\sum_{\kappa=1}^{\infty}\beta_{\kappa\mu}L_{\varphi_\kappa}(x)]\Psi_\mu^*=\sum_{j=1}^{\infty}[\sum_{\kappa=1}^{\infty}\sum_{\mu=1}^{\infty}\beta_{\kappa\mu}\bar{\beta}_{j\mu}L_{\varphi_\kappa}(x)]\psi_j,$$

where, by hypotheses, $\sum_{\mu=1}^{\infty}\bar{\beta}_{\mu\kappa}\beta_{\mu j}=\sum_{\mu=1}^{\infty}\beta_{\kappa\mu}\bar{\beta}_{j\mu}$ for every pair of $\kappa, j=1, 2, 3, \cdots$. These results lead us to the conclusion that $NN^*x=N^*Nx$ for every $x\in\mathfrak{H}$. Thus N is a normal operator in $\mathfrak{H}$.

Furthermore the hypothesis $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2\neq|\beta_{\mu\mu}|^2$ for $\mu=1, 2, 3, \cdots$ enables us to assert that N has no eigenvalue other than λ_ν, $\nu=1, 2, 3, \cdots$, as can be seen by following the argument used in the proof of the case where (β_{ij}) is an infinite unitary matrix with $|\beta_{jj}|\neq 1, j=1, 2, 3, \cdots$ [cf. Proc. Japan Acad., Vol. 37, p. 617 (1961)].

With these results the proof of the theorem is complete.

Next we shall consider the question as to whether conversely any bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $\mathfrak{H}$ can be expressed by such a functional-representation as was defined by the right-hand member of (1).

Theorem C. Let N be a bounded normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be its point spectrum (inclusive of the multiplicity of each eigenvalue); let φ_ν be a normalized eigenelement of N corresponding to the eigenvalue λ_ν for any value of $\nu=1, 2, 3, \cdots$; let $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ be an incomplete orthonormal set orthogonal to $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ such that a complete orthonormal system in $\mathfrak{H}$ can be constructed by these two orthonormal sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$; let c be a non-zero complex constant; and let $\Psi_\mu=\sum_{j=1}^{\infty}\beta_{\mu j}\psi_j$ where $\beta_{\mu j}=(N\psi_\mu, \psi_j)/c$ for every pair of $\mu, j=1, 2, 3, \cdots$. Then N is expressed in the form

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu},$$

and both $\sum_{j=1}^{\infty}|\beta_{j\mu}|^2$ and $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2$ never exceed $||N||^2/|c|^2$ for every value of $\mu=1, 2, 3, \cdots$. Furthermore, not only (β_{ij}) is a normal matrix with $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2\neq|\beta_{\mu\mu}|^2$, $\mu=1, 2, 3, \cdots$, but also the operator B associated with (β_{ij}) is a bounded (normal) operator in l_2.

Proof. Since, by hypotheses, a complete orthonormal system in $\mathfrak{H}$ can be constructed by the mutually orthogonal sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$ and since φ_ν is a normalized eigenelement of N^* corresponding to the eigenvalue $\bar{\lambda}_\nu$, we have

$$Nx=\sum_{\nu=1}^{\infty}(Nx, \varphi_\nu)\varphi_\nu+\sum_{j=1}^{\infty}(Nx, \psi_j)\psi_j$$
$$=\sum_{\nu=1}^{\infty}\lambda_\nu(x, \varphi_\nu)\varphi_\nu+\sum_{j=1}^{\infty}(x, N^*\psi_j)\psi_j$$

for every $x\in\mathfrak{H}$. Since, moreover, $(N^*\psi_j, \varphi_\nu)=(\psi_j, N\varphi_\nu)=\bar{\lambda}_\nu(\psi_j, \varphi_\nu)=0$,

$$(x, N^*\psi_j) = \sum_{\nu=1}^{\infty}(x, \varphi_\nu)(\overline{N^*\psi_j, \varphi_\nu}) + \sum_{\mu=1}^{\infty}(x, \psi_\mu)(\overline{N^*\psi_j, \psi_\mu})$$
$$= c\sum_{\mu=1}^{\infty}\beta_{\mu j}(x, \psi_\mu),$$

so that

$$(2) \qquad Nx = \sum_{\nu=1}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\varphi_\nu + c\sum_{j=1}^{\infty}[\sum_{\mu=1}^{\infty}\beta_{\mu j}L_{\phi_\mu}(x)]\psi_j.$$

On the other hand, by reference to the relations $(N^*\psi_\mu, \varphi_j)=0$, $(\mu, j=1, 2, 3, \cdots)$, we have

$$\sum_{j=1}^{\infty}|\beta_{j\mu}|^2 = \frac{1}{|c|^2}\sum_{j=1}^{\infty}|(\overline{N^*\psi_\mu, \psi_j})|^2 = \frac{||N^*\psi_\mu||^2}{|c|^2} \leqq \frac{||N||^2}{|c|^2}$$

and similarly $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2 \leqq ||N||^2/|c|^2$ for every $\mu=1, 2, 3, \cdots$. Accordingly $\sum_{j=1}^{\infty}\beta_{\mu j}\psi_j \in \mathfrak{H}$ and

$$|\sum_{\mu=1}^{\infty}\beta_{\mu j}L_{\phi_\mu}(x)|^2 \leqq \{\sum_{\mu=1}^{\infty}|\beta_{\mu j}L_{\phi_\mu}(x)|\}^2$$
$$\leqq \frac{||N||^2||x||^2}{|c|^2},$$

which implies that $\sum_{j=1}^{\infty}[\sum_{\mu=1}^{\infty}\beta_{\mu j}L_{\phi_\mu}(x)]\psi_j$ is in fact an element belonging to $\mathfrak{H}$. From (2) we thus obtain the relation

$$Nx = \sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu \otimes L_{\varphi_\nu}(x) + c\sum_{\mu=1}^{\infty}\Psi_\mu \otimes L_{\phi_\mu}(x)$$

holding for every $x \in \mathfrak{H}$, so that

$$N = \sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu \otimes L_{\varphi_\nu} + c\sum_{\mu=1}^{\infty}\Psi_\mu \otimes L_{\phi_\mu},$$

as we wished to prove.

By making use of the relations $(N\psi_i, \varphi_\nu)=0$, $(i, \nu=1, 2, 3, \cdots)$, we next have

$$\sum_{\mu=1}^{\infty}\beta_{i\mu}\bar{\beta}_{j\mu} = \frac{1}{|c|^2}\sum_{\mu=1}^{\infty}(N\psi_i, \psi_\mu)(\overline{N\psi_j, \psi_\mu})$$
$$= \frac{(N\psi_i, N\psi_j)}{|c|^2}$$
$$= \frac{(N^*N\psi_i, \psi_j)}{|c|^2}$$

and similarly

$$\sum_{\mu=1}^{\infty}\bar{\beta}_{\mu i}\beta_{\mu j} = \frac{(NN^*\psi_i, \psi_j)}{|c|^2}.$$

Since, in addition, N is bounded and normal by hypotheses, $N^*N = NN^*$ in $\mathfrak{H}$ and hence the just established relations permit us to conclude that $\sum_{\mu=1}^{\infty}\beta_{i\mu}\bar{\beta}_{j\mu} = \sum_{\mu=1}^{\infty}\bar{\beta}_{\mu i}\beta_{\mu j}$. This last result shows that the matrix (β_{ij}) is normal. We must here prove that $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2 \neq |\beta_{\mu\mu}|^2$ for every value of $\mu=1, 2, 3, \cdots$. However this is a direct consequence of the hypothesis that the eigenspace of N is determined by the set $\{\varphi_\nu\}$: for, if $\sum_{j=1}^{\infty}|\beta_{\mu j}|^2=|\beta_{\mu\mu}|^2$ for $\mu=\kappa$, we would have

$$\begin{aligned} N\psi_\kappa &= \sum_{\nu=1}^{\infty}\lambda_\nu(\psi_\kappa, \varphi_\nu)\varphi_\nu + c\sum_{\mu=1}^{\infty}\Big[(\psi_\kappa, \psi_\mu)\sum_{j=1}^{\infty}\beta_{\mu j}\psi_j\Big] \\ &= \beta_{\kappa\kappa}\psi_\kappa, \end{aligned}$$

contrary to that hypothesis.

Lastly it remains only to prove that the operator B associated with the matrix (β_{ij}) is bounded in l_2. Let now $\tilde{x}=(x_1, x_2, x_3, \cdots)\in l_2$, and let $f=\sum_{\mu=1}^{\infty}\bar{x}_\mu\psi_\mu$. Then, since $\sum_{\mu=1}^{\infty}|\bar{x}_\mu|^2<\infty, f$ is an element belonging to the subspace determined by the set $\{\psi_\mu\}$ and hence $\bar{x}_\mu=(f, \psi_\mu)$. In consequence, by applying again the relations $(N\psi_j, \varphi_\nu)=(N^*f, \varphi_\nu)=0$, $(j, \nu=1, 2, 3, \cdots)$, and the hypothesis that a complete orthonormal system in $\mathfrak{H}$ is constructed by the two sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$, we obtain

$$\begin{aligned} \|B\tilde{x}\|^2 &= \sum_{j=1}^{\infty}\Big|\sum_{\mu=1}^{\infty}\beta_{j\mu}x_\mu\Big|^2 \\ &= \frac{\sum_{j=1}^{\infty}\Big|\sum_{\mu=1}^{\infty}(N\psi_j, \psi_\mu)\overline{(f, \psi_\mu)}\Big|^2}{|c|^2} \\ &= \frac{\sum_{j=1}^{\infty}|(N\psi_j, f)|^2}{|c|^2} \\ &= \frac{\sum_{j=1}^{\infty}|\overline{(N^*f, \psi_j)}|^2}{|c|^2} \\ &= \frac{\|N^*f\|^2}{|c|^2} \\ &\leqq \frac{\|N\|^2\|\tilde{x}\|^2}{|c|^2}. \end{aligned}$$

This final inequality shows that B is a bounded operator in l_2, as we were to prove.

The theorem has thus been proved.