160. *The Asymptotic Behaviour of the Solution of a Semi-linear Partial Differential Equation Related to an Active Pulse Transmission Line*

By Masaya YAMAGUTI

(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1963)

1. **Introduction.** J. Nagumo [1] proposed as active pulse transmission line simulating an animal nerve axon. The equation of propagation of his line is the following:

$$(1)\qquad \frac{\partial^2 u}{\partial t^2}=\frac{\partial^3 u}{\partial x^2 \partial t}-\mu(1-u+\varepsilon u^2)\frac{\partial u}{\partial t}-u \qquad \begin{matrix}\mu>0,\ \varepsilon>0\\ x>0,\ t>0\end{matrix}$$

with the boundary data;

$$(2)\qquad \begin{cases} u(x,0)=0 & (x\geq 0)\\ u_t(x,0)=0 & (x\geq 0)\\ u(0,t)=\psi(t) & (t\geq 0),\ \psi(t)\equiv 0 \quad \text{for} \quad t\geq t_0. \end{cases}$$

In this note, we consider some asymptotic behaviours of the solution for the equation of related type with the same boundary data: Our equation is the following:

$$(3)\qquad \frac{\partial^2 u}{\partial t^2}=\frac{\partial^3 u}{\partial x^2 \partial t}-f'(u)\frac{\partial u}{\partial t}-g(u).$$

At first, we remark that the existence of global solutions for this problem (3) with boundary data (2) where $\psi(t)\in C^2$ is assumed was completely proved by R. Arima and Y. Hasegawa [2] under the conditions:

$$(4)\qquad \begin{cases} -K_1\leq f'(u)\leq K_0(u^2+1),\\ |g(u)|\leq K_2(u^2+|u|),\\ G(u)=\int_0^u\{-g(z)\}dz\leq K_3u^2, \end{cases}$$

$$g(u),\ f'(u)\in C^1.$$

Throughout this paper, we always assume that $f'(u)$, $g(u)$ satisfy this condition (4).

Our results are divided into two parts. The one is the case $g(u)\equiv u$, the other is the case $g(u)\equiv 0$. For the first case, we can prove that any solution $u(x,t)$ tends uniformly to zero, when t tends to $+\infty$, under the additional condition (5), which corresponds to the limitation $\varepsilon>\frac{3}{16}$ in (1). For the second case we can show the existence of a threshold value for the boundary data (Prop. 3) and a sort of asymptotic value under another additional conditions (Prop. 4), (9), (11), which is independent of (5).

We remark also that the summability in x, of $u(x,t)^2$ and $u_x(x,t)^2$, which is shown in [2], will play an important role in our proofs.

2. **The first case.** We assume that $g(u)\equiv u$ and we also assume the following condition (5) (which will be imposed only in this section).

There exists a positive constant c such that

$$(5)\qquad uf(u)\geq cu^2 \text{ where } f(u)=\int_0^u f'(z)dz.$$

Then we have

PROPOSITION 1. *For arbitrary given data $\psi(t)\in C^2$ appeared in (2), the solution of (3) tends uniformly to zero when t tends to $+\infty$ under the condition (5).*

PROOF. We transform the equation (3) to a system of equations by integration with respect to t and putting $\int_0^t u(x,\tau)d\tau=w(x,t)$,

$$(6)\qquad \begin{cases} u_t=u_{xx}-f(u)-w \\ w_t=u. \end{cases}$$

We use the following energy form to obtain an energy inequality, here we denote by $F(u)$ the primitive function of $f(u)$ taking $F(0)=0$,

$$(7)\qquad E(t)=\int_0^\infty\left[\frac{u^2}{2}+\frac{u_t^2}{2}+k\frac{u_x^2}{2}+kF(u)+k\frac{(u+w)^2}{2}+L\frac{u^2+w^2}{2}\right]dx,$$

where $k>K_1$ and L is a positive constant so large that it satisfies $\frac{k}{k+L}<c'<c$. Differentiating (7) with respect to t, and by the integration by parts, we obtain

$$(8)\qquad E'(t)=-\int_0^\infty[u_{xt}^2+(f'(u)+k)u_t^2+(k+L)u_x^2+(k+L)uf(u)-ku^2]\,dx$$
$$\leq 0 \text{ for } t\geq t_0,$$

where t_0 is a constant such that $\psi(t)\not\equiv 0$ in $0\leq t\leq t_0$ but $\psi(t)\equiv 0$ in $t\geq t_0$.

We can conclude from the differential inequality (8), following facts are finite:

a) $E(t)$ is non-increasing in t for $t\geq t_0$.

b) $0\leq E(t)\leq E(t_0)$. Consequently $\lim_{t\to+\infty} E(t)$ exists.

c) Integrals:

$\int_0^\infty \frac{u^2}{2}dx, \int_0^\infty \frac{u_t^2}{2}dx, \int_0^\infty \frac{w^2}{2}dx, \int_0^\infty u_x^2dx$ are bounded for $t\geq t_0$.

d) Integrals:

$$\int_{t_0}^\infty\int_0^\infty \frac{u^2}{2}dxd\tau, \int_{t_0}^\infty\int_0^\infty \frac{u_t^2}{2}dxd\tau, \int_{t_0}^\infty\int_0^\infty \frac{u_x^2}{2}dxd\tau, \int_{t_0}^\infty\int_0^\infty \frac{u_{xt}^2}{2}dxd\tau.$$

If we put $\varphi(t)=\int_0^\infty\left[\frac{u^2}{2}+\frac{u_x^2}{2}\right]dx$, we can show $\varphi(t)\to 0(t\to+\infty)$. Because,

$$|\varphi(t)-\varphi(t')|=\left|\int_{t'}^{t}\varphi'(\tau)d\tau\right|=\int_0^\infty\int_{t'}^{t}[uu_t+u_xu_{xt}]\,dx\,d\tau$$

$$\leq\left[\int_{t'}^{t}\int_0^\infty u^2dxd\tau\right]^{\frac{1}{2}}\left[\int_{t'}^{t}\int_0^\infty u_t^2dxd\tau\right]^{\frac{1}{2}}+\left[\int_{t'}^{t}\int_0^\infty u\,dx\,d\tau\right]^{\frac{1}{2}}\left[\int_{t'}^{t}\int_0^\infty u_{xt}^2dx\,d\tau\right]^{\frac{1}{2}}.$$

By d) above, we can find a constant T for arbitrary given $\varepsilon>0$, such that $|\varphi(t)-\varphi(t')|<\varepsilon$ for $t, t'>T$. Then $\lim_{t\to+\infty}\varphi(t)$ exist and by the summability we see $\lim_{t\to+\infty}\varphi(t)=0$. Consequently we can prove that $\max_{0\leq x<+\infty}|u(x,t)|\to 0$ for $t\to+\infty$, by the Sobolev's lemma.

3. The second case ($g(u)\equiv 0$). At first we mention some additional conditions for this case:

$$(9)\qquad \begin{cases} f(u)<0 \quad (u<0), & f(u)<0 \quad (a<u<b),\\ f(u)>0 \quad (0<u<a), & f(u)>0 \quad (b>u),\end{cases}$$

here a and b are two distinct constants.

Under this condition (9), we can prove a generalized maximum-minimum principle for the solution of (3) for $g(u)=0$. That is

PROPOSITION 2. *Under the condition* (9), *if B_0 is a constant greater than b, then $u(x,t_0)<B_0$ implies always $u(x,t)<B_0$ for $t\geq t_0$, and $u(x,t_0)\geq 0$ implies $u(x,t)\geq 0$ for $t>t_0$.*

PROOF. If there is a point (x_1,t_1) where $u(x_1,t_1)=B_0$, then we can consider the set E of (x,t) $t>t_0$, $0\leq x<+\infty$ such that $u(x,t)=B_0$. We can prove by the contradiction that there is a positive distance $\delta>0$ between the set E and the half straight line $t=t_0$ $0\leq x<+\infty$. If not, there should be a sequence of points (ξ_n,τ_n) which tends to one point of this half line or $+\infty$ point of this half line, it signifies that there exists a point (ξ,t_0) where $u(\xi,t_0)=B_0$ or a sequence of points (ξ_n',t_0) $(\xi_n'\to+\infty)$ where $u(\xi_n',t_0)\geqq\frac{B_0}{2}$, by the fact that $u_t(x,t)$ is bounded for $0\leq t\leq T$. (T is some constant $\geq t_0$.) This latter case contradicts to the fact that $u(x,t_0)^2$ and $u_x(x,t_0)^2$ are summable in $0\leq x<+\infty$ [2].

Therefore we find a point (x_1,t_1) where $u(x_1,t_1)=B_0$, $u(x,t)<B_0$ $(t_0\leq t<t_1,\ 0\leq x<+\infty)$ and $u(x,t_1)<B$ $(x<x_1)$. Because $u(x,t)$ is a solution of (2), (3), $u_t=u_{xx}-f(u)$, $u_{xx}(x_1,t_1)\leq 0$ and $f(B_0)=f(u(x_1,t_1))$. Consequently $u_t(x_1,t_1)<0$, this means that there exists a point (x_1,t_2) $(t_2<t_1)$ such that $u(x_1,t_2)>u(x_1,t_1)=B_0$. This is a contradiction. The same argument shows that $u(x,t_0)\geq 0$ implies $u(x,t)\geq 0$ for $t\geq t_0$.

We add still one additional assumption:

(10) There exist two positive constants c_1 and a_1 such that

$$uf(u)\geq c_1(u^2+F(u)) \text{ for } 0\leq u\leq a_1<a.$$

PROPOSITION 3. *Under the assumptions* (9) *and* (10), *if $0\leq u(x,t_0)<a_1$, $u(x,t)$ tends exponentially to zero when t tends to $+\infty$, in the maximum norm.* Before entering into the proof, we remark that the

same discussion in Proposition 2 shows that if $u(x, t_0) < a_1$, $u(x, t)$ remains always less than a_1 for $t \geq t_0$.

PROOF. We consider an energy form

$$E_1(t) = \int_0^\infty \left[\frac{u^2}{2} + \frac{u_x^2}{2} + F(u)\right] dx,$$

Differentiating with respect to t, we have

$$E_1'(t) = -\int_0^\infty [u_x^2 + uf(u) + u_t^2]\, dx$$

$$< -\int_0^\infty [u_x^2 + c_1(u^2 + F(u))]\, dx < -c_2 E_1(t), \ c_2 > 0.$$

Consequently, we obtain

$$E_1(t) \leq E_1(t_0) e^{-c_2 t}.$$

By the Sobolev's lemma, we conclude that $u(x, t)$ tends exponentially in the sense of uniform maximum norm.

Finally we assume the additional condition:

$$(11) \qquad \begin{cases} f'(u) > 0 & (u < \alpha), \quad f(u)^2 \geq c_0' F(u) \quad (u \leq 0) \\ f'(u) < 0 & (\alpha < u < \beta), \\ f'(u) > 0 & (\beta < u), \end{cases}$$

where $0 < \alpha < \beta < b < B$. We denote B the point such that $F(B) = 0$, $F(u) > 0$ for $u > B$.

PROPOSITION 4. *Denoting B_1 a constant greater than B, and M_t the set of x, $(0 \leq x < +\infty)$ such that $u(x, t) \geq B_1$, then the measure of the set M_t tends exponentially to zero when t tends to $+\infty$ under conditions* (9) *and* (11). *Moreover under same condition, the integral $\int_{M_t} u^2(x, t)\, dx$ tends also exponentially to 0 when t tends to $+\infty$.*

PROOF. We can construct a 2 times differentiable function $\Phi(u)$ as follows:

$$\begin{cases} \Phi(u) \equiv F(u) & u \leq 0 \\ \Phi(u) \equiv 0 & 0 \leq u \leq B_2 \quad (b < B_2 < B) \\ \Phi(u) \equiv X(u) & u \geq B_2 \end{cases}$$

where $X(u)$ satisfies following conditions:

$$(12) \qquad \begin{cases} X'(u) \geq \dfrac{1}{f(B_2)} X(u), \ X(u) > 0, \ X''(u) > 0 \ \text{for } u > B_2 \\ X'(B_2) = X(B_2) = 0 \\ \text{there exists a positive constant } c_3 \text{ such that } X(u) \geq c_3 u^2 \text{ for } u \geq B_1. \end{cases}$$

In fact, taking $\varphi(u)$ such that satisfies $\varphi(u)$, $\varphi'(u)$, $\varphi''(u) > 0$ for $u > B_2$ $\varphi'(B_2) = \varphi(B_2) = 0$, and $\varphi(u) \geq c_4 u^2$ for $u \geq B_1$, and setting $X(u) = e^{\frac{u}{f(B_2)}} \varphi(u)$ we see $X(u)$ satisfies (12). This means that $\Phi(u)$ satisfies always $\Phi'(u) f(u) \geq c_0' \Phi(u)$.

Now we use the following a new energy form

(13) $$E_2(t)=\int_0^\infty \Phi(u)dx$$

(the summability of $\Phi(u)$ is evident by the fact that u^2 and u_x^2 are summable). Differentiating with respect to t, we have

$$E_2'(t)=\int_0^\infty \Phi'(u)u_t dx$$
$$=-\int_0^\infty \Phi''(u)u_x^2 dx-\int_0^\infty \Phi'(u)f(u)dx.$$

By the condition (12) and remarking that $f(u)>f(B_2)$ for $u>B_2$, we have

$$E_2'(t)\leq -c_0'E_2(t).$$

That is $$E_2(t)\leq E_2(0)e^{-c_0't}.$$

It follows that $$c_4\int_{M_t} u^2 dx\leq \int_0^\infty \Phi(u)dx\leq E_2(0)e^{-c_0't}.$$

Relating Proposition 3, we mention a remark.

REMARK. Considering an another energy form which is not positive definite, we can show the existence of a solution of (3) in the case $g(u)\equiv 0$, which does not tends uniformly to zero when t tends to $+\infty$. In fact, taking

$$E_3(t)=\int_0^\infty \left[\frac{u_x^2}{2}+F(u)\right]dx,$$

if $u(x, t_0)$ satisfies

$$\int_0^\infty \left[\frac{u_x^2(x, t_0)}{2}+F(u(x, t_0))\right]dx<0,\ u(x, t_0)\geq 0,$$

then $u(x, t)$ does not tend to zero uniformly; because, if not, for $t>T_1$ (sufficiently large)

$$|u(x, t)|<\varepsilon$$

for given $\varepsilon<a_1<a$, then by the same discussion in Proposition 2, we have

$$0\leq u(x, t)<a_1.$$

It signifies that $E_3(t)=\int_0^\infty \left[\frac{u_x^2}{2}+F(u)\right]dx\geq 0$ by the condition (9). That is a contradiction because we can show always that

$$E_3'(t)=-\int_0^\infty u_t^2 dx\leq 0\ \ (t\geq t_0).$$

References

[1] J. Nagumo, S. Arimoto, and S. Yoshizawa: An active pulse transmission lines simulating nerve axon, Proceedings of the IRE, **50** (10), 2061–2070 (1962).

[2] R. Arima, and Yōjirō Hasegawa: On global solutions for mixed problem of a semi-linear differential equation, Proc Japan Acad., **39**, 721–725 (1963).