159. *On Global Solutions for Mixed Problem of a Semi-linear Differential Equation*

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1. **Introduction.** Let us consider the equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^3 u}{\partial t \partial x^2}=f(u)\frac{\partial u}{\partial t}+g(u) \tag{1.1}$$

in the half space $\Omega=\{(x, t);\ x, t>0\}$.

Such an equation was considered by J. Nagumo as a model of the neuron.[1] Let us limit the behaviour of the function f and g in (1.1) as follows:

$$\begin{cases} f,\ g \in C^1,\ g(0)=0,\ -K_0(u^2+1)\leq f(u) \leq K_1, \\ |g(u)|\leq K_2(u^2+|u|) \text{ and moreover} \\ G(u)\equiv \int_0^u g(z)dz \leq K_3 u^2 \end{cases} \tag{1.2}$$

where K_0, K_1, K_2, K_3 are positive constants.

Now the initial and boundary data are given as follows with the compatibility conditions

$$\begin{cases} u(x, 0)=u_0(x) \in \mathscr{B}_+^2 \cap \mathscr{D}_{L^2+}^1 & \text{for } x\geq 0, \\ u_t(x, 0)=u_1(x) \in \mathscr{B}_+^2 \cap \mathscr{D}_{L^2+}^1 & \text{for } x\geq 0, \\ u(0, t)=\psi(t) \in C^2 \quad \text{for } t\geq 0, \end{cases} \tag{1.3}$$

$$\begin{cases} u_0(0)=\psi(0),\ u_1(0)=\psi'(0) \\ \psi''(0)-u_1''(0)=f(\psi(0))\psi'(0)+g(\psi(0)). \end{cases} \tag{1.4}$$

Then we can prove the following:

THEOREM 1. *There exists a unique solution $u(x, t)$ in Ω and* $u(x, t)$, $u_t(x, t) \in (\mathscr{B}_+^2 \cap \mathscr{D}_{L^2+}^1)$ $[0, T]$. (Throughout this paper, we use the following notation. Let E be a topological vector space. $f(x, t)$, or simply $f(t)$ belongs to $E[0, T]$, if $f(x, t)$ is a continuous function in $t \in [0, T]$ with values in E. $\mathscr{B}_+^k$ is the topological vector space of uniformly continuous and bounded functions in $(0, \infty)$ together with their derivatives of order up to k. If we consider square integrable functions instead of uniformly continuous and bounded functions, we have $\mathscr{D}_{L^2+}^k$.)

To prove this theorem, we should obtain a priori estimates of solution and local existence theorem adapted to the step by step continuation.

2. **Local existence theorem.** *Let us consider the problem in $0\leq t\leq T$, then there exists a function $\varphi(\xi_1, \xi_2, \xi_3)$, positive and non-increasing in each argument, such that:*

Let t_0 be any point in the interval, then for initial data u_0, u_1 at $t=t_0$ and boundary data ψ, there exists the solution in $t_0 \leq t \leq t_0+h$, where $h=\varphi(\sup_{0\leq x<+\infty}|u_0(x)|, \sup_{0\leq x<+\infty}|u_1(x)|, \max_{0\leq t\leq T}|\psi'(t)|)$.

Let's remark that

$$G(x, \xi, t-\tau)=E(x-\xi, t-\tau)-E(x+\xi, t-\tau)$$

in the Green kernel of the heat equation in the half space, where

$$E(x, t)=\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right).$$

The kernel has the following property;

$$G(x, \xi, t)\geq 0, \quad \int_0^\infty G(x, \xi, t)\, d\xi \leq 1.$$

In fact,

$$\int_0^\infty G(x, \xi, t)\, d\xi \leq \int_0^\infty E(x-\xi, t)\, d\xi$$

$$=\int_0^\infty \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{(x-\xi)^2}{4t}\right)d\xi,$$

changing the variable ξ to $\xi'=\frac{x-\xi}{2\sqrt{t}}$,

$$=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2t}}^\infty \exp\{-\xi'^2\}\, d\xi' \leq 1.$$

Let us put $t_0=0$ without loss of generality. From (1.1), (1.3), we have the following integro-differential equation:

$$u(x, t)=\phi(x, t)+\int_0^t d\tau \int_0^\tau ds \int_0^\infty G(x, \xi, \tau-s) \times \left\{f(u(\xi, \tau))\frac{\partial u}{\partial s}(\xi, s)+g(u(\xi, s))\right\}d\xi, \tag{2.1}$$

where

$$\phi(x, t)=u_0(x)-2\int_0^t E_x(x, t-\tau)\psi(\tau)\, d\tau + \int_0^t d\tau \int_0^\infty G(x, \xi, \tau)\, u_1(\xi)\, d\xi, \tag{2.2}$$

which satisfies the equation $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^3\phi}{\partial t\partial x^2}=0$ with the condition (1.3).

Put

$$u_0(x, t)=\phi(x, t)$$

$$u_i(x, t)=\phi(x, t)+\int_0^t d\tau \int_0^\tau ds \int_0^\infty G(x, \xi, t-s) \times \left\{f(u_{i-1})\frac{\partial u_{i-1}}{\partial s}+g(u_{i-1})\right\} d\xi$$

$$(i=1, 2, 3, \cdots).$$

Then we have easily

$$|u_i(x,t)| \leq |\phi(x,t)| + \frac{1}{2} \sup_{\substack{0\leq x<+\infty \\ 0\leq s\leq t}} \left| f(u_{i-1})\frac{\partial u_{i-1}}{\partial s} + g(u_{i-1}) \right|$$

and

$$\left|\frac{\partial u_i}{\partial t}(x,t)\right| \leq \left|\frac{\partial}{\partial t}\phi(x,t)\right| + t \sup_{\substack{0\leq x<+\infty \\ 0\leq s\leq t}} \left| f(u_{i-1})\frac{\partial u_{i-1}}{\partial s} + g(u_{i-1}) \right|.$$

If we take

$$h = \min\left[\frac{N-C}{MN+M}, 1, T\right]$$

where $C = \sup_{\substack{0\leq x<+\infty \\ 0\leq t\leq T}} \left|\frac{\partial}{\partial t}\phi(x,t)\right|$, N is any number greater than C and $M = \max_{|u|\leq N}(|f(u)|, |g(u)|)$, then we have $|u_i(x,t)|$, $\left|\frac{\partial u_i}{\partial t}(x,t)\right| \leq N$ $(i=0,1,2,\cdots)$ in $0\leq x<+\infty$, $0\leq t\leq h$.

At first $\phi \in (\mathscr{B}^2_+ \cap \mathscr{D}^1_{L^2})[0, T]$.

In fact

$$E_x(x, t-\tau) \leq 0, \quad 0 \leq \int_0^t -E_x(x, t-\tau)\,d\tau \leq \frac{1}{2} \text{ for all } t>0,\ x>0.$$

It follows $\phi \in (\mathscr{B}^0_+ \cap L^2_+)[0, T]$.

The proof is similar for the derivatives of ϕ, because, remarking $E_t = E_{xx}$, we have

$$\frac{\partial}{\partial x}\int_0^t E_x(x, t-\tau)\psi(\tau)\,d\tau = \frac{\partial}{\partial t}\int_0^t E(x, t-\tau)\psi(\tau)\,d\tau$$
$$= \int_0^t E(x, t-\tau)\psi'(\tau)\,d\tau,$$

and

$$\frac{\partial}{\partial t}\int_0^t E_x(x, t-\tau)\psi(\tau)\,d\tau = \int_0^t E_x(x, t-\tau)\psi'(\tau)\,d\tau.$$

As easily seen, the sequence $\{u_n\}$ is convergent in $\mathscr{B}^0_+[0, h]$ with the analogous discussions, the limit function u belongs to $(\mathscr{B}^2_+ \cap \mathscr{D}^1_{L^2})$ $[0, h]$ and satisfies all the required properties in Theorem 1.

3. **A priori estimates.** In the previous section, we obtained the local existence theorem. In this section, we will show that $|u(x,t)|$ and $\left|\frac{\partial u}{\partial t}(x,t)\right|$ have *a priori* bounds in $0\leq t\leq T$, where T is any positive number. This shows that we can choose the same number h in $0\leq t\leq T$ in the local existence theorem. It follows that there exists the solution in $0\leq t\leq h$ at first, then we can find the solution in $0\leq t\leq 2h$ and thus, step by step, we have the solution in $0\leq t\leq T$. If we assume $|u(x,t)|$ has an *a priori* bound, we have easily an *a priori* bound also for $\left|\frac{\partial u}{\partial t}(x,t)\right|$, by using the equalities (2.1) and (2.2). So we have only to show $u(x,t)$ has an *a priori* bound.

Let $u(x, t)$ be a solution satisfying (1.1) and (1.3), and

$$u(x, t),\ u_t(x, t) \in \mathcal{D}^1_{L^2+}[0, T].$$

Put $v(x, t)=u(x, t)-\phi(x, t)$, where $\phi(x, t)$ is defined by (2.1), then

$$\begin{aligned}&v,\ v_t \in \mathcal{D}^1_{L^2}[0, T]\\ &v_{tt}-v_{txx}=f(v+\phi)(v_t+\phi_t)+g(v+\phi),\\ &v(x, 0)=v_t(x, 0)=0,\\ &v(0, t)=0.\end{aligned}$$

Now, let us consider the energy form on $v(x, t)$;

$$E(t)=\int_0^\infty \left[\frac{1}{2}(v_t^2+v_x^2+Cv^2)-G(v+\phi)\right]dx,$$

where $C=1+4K_3$, then, taking the derivative with respect to t, we have

$$\begin{aligned}E'(t)&=\int_0^\infty \{v_t v_{tt}+v_x v_{xt}+Cvv_t-g(v+\phi)(v_t+\phi_t)\}\,dx\\ &=\int_0^\infty \{v_t[v_{txx}+f(v+\phi)(v_t+\phi_t)+g(v+\phi)]\\ &\quad+v_x v_{tx}+Cvv_t-g(v+\phi)(v_t+\phi_t)\}\,dx.\end{aligned}$$

Remarking

$$\int_0^\infty v_t v_{txx}dx=-\int_0^\infty v_{tx}^2 dx$$

and

$$(v_t+\phi_t)v_t=\left(v_t+\frac{1}{2}\phi_t\right)^2-\frac{\phi_t^2}{4},$$

and by the assumption (1.2) for f and g, we have

$$E'(t)\leq C_1E(t)+C_2 \qquad 0\leq t\leq T$$

where C_1, C_2 are positive constants.

More pricisely they depend only on

$$\gamma=\max_{0\leq t\leq T}\Psi(t),\ \Psi(t)=\max\left(|\phi(t)|,\ |\phi'(t)|,\ \|\phi(t)\|_{L^2},\ \|\phi'(t)\|_{L^2}\right).^{*)}$$

Therefore $E(t)$, $0\leq t\leq T$, has an *a priori* bound. It follows, by using Sobolev's lemma, that $v(t)$, therefore $u(t)$ has an *a priori* bound.

Thus we have obtained

PROPOSITION. *Under the assumption in Theorem* 1, $|u(x, t)|$ *and* $\left|\frac{\partial u}{\partial t}(x, t)\right|$ *have a priori bounds in any finite interval of t.*

REMARK. We can consider the strictly analogous problem in the three dimensional space of x and we have analogous results.

$$(1.1)' \qquad \frac{\partial^2}{\partial t^2}u-\frac{\partial}{\partial t}\Delta u=f(u)\frac{\partial}{\partial t}u+g(u),\ \Delta\equiv\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$$

in $\Omega=\{(x_1, x_2, x_3, t);\ 0<t,\ 0<x_1<+\infty,\ -\infty<x_2, x_3<+\infty\}$ under the condition (1.2).

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*) $|\varphi(t)|$ stands for $\sup_x|\varphi(x, t)|$. $\|\varphi(t)\|_{L^2}$ stands for $\|\varphi(x, t)\|_{L^2(R_x)}$.

The initial-boundary data are given as follows:

$$(1.3)' \quad \begin{cases} u=u_0,\ \dfrac{\partial u}{\partial t}=u_1 & (t=0) \\ u=\psi & (x_1=0) \end{cases}$$

$$(1.4)' \quad \begin{cases} u_0=\psi,\ u_1=\psi_t \\ \psi_{tt}-\Delta u_1=f(\psi)\psi'+g(\psi) \end{cases} \quad \begin{pmatrix} t=0 \\ x_1=0 \end{pmatrix}$$

where

$u_0(x_1, x_2, x_3),\ u(x_1, x_2, x_3)\in \mathscr{B}^1_+ \cap \mathscr{D}^2_{L^2+}$; moreover,
$u_{0x_ix_i},\ u_{1x_ix_i}\in \mathscr{B}^0_+ \quad (i=1, 2, 3)$
$\psi_t(x_2, x_3, t)\in (\mathscr{B}^2_+ \cap \mathscr{D}^2_{L^2})[0, T]$
$\psi_{tt}(x_2, x_3, t)\in (\mathscr{B}^0_+ \cap L^2)[0, T]$.

Then we have

THEOREM 2. *There exists a unique solution $u(x, t)$ in Ω and*
$u(x, t),\ u_t(x, t)\in (\mathscr{B}^1_+ \cap \mathscr{D}^2_{L^2+})[0, T]$,
$u_{x_ix_i}(x, t),\ u_{tx_ix_i}(x, t)\in \mathscr{B}^0_+[0, T]$.

The proof of the local existence theorem is almost analogous to Theorem 1, and for *a priori* estimates, we need consider two energy forms:

$$E_1(t)=\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \left\{\frac{1}{2}[v_t^2+\sum_i v_{x_i}^2+Cv^2]-G(v+\phi)\right\}dx_1dx_2dx_3,$$

$$E_2(t)=\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{1}{2}[\sum_i v_{tx_i}^2+\sum_{i,j} v_{x_ix_j}^2]dx_1dx_2dx_3.$$

Reference

[1] J. Nagumo, S. Arimoto, and S. Yoshizawa: An active pulse transmission line simulating nerve axon. Proceedings of the IRE, **50**(10), 2061–2070 (1962.)