157. On the Immersibility of almost Parallelizable Manifolds

By Tuyosi WATABE and Tunehisa HIROSE
Department of Mathematics, Niigata University
(Comm. by Zyoiti SUETUNA, M.J.A., Dec. 12, 1963)

1. M. W. Hirsch has shown that an almost parallelizable n-manifold is immersible in the Euclidean $(n+k)$-space R^{n+k} if $n<2k$ [1]. He has proved the result by making use of a result due to M. Kervaire that the Smale invariant of an immersion of n-sphere in R^{n+k} vanishes if $n\leqq 2k-2$ [4]. In this paper we shall prove the following*)

Proposition 1. An almost parallelizable n-manifold is immersible in R^{n+1} if $n\not\equiv 0$ (mod 4).

Proposition 2. If $n\equiv 0$ (mod 4), an almost parallelizable n-manifold is in general not immersible in R^{n+1}. In particular, Hirsch's result is best possible for $n=4$. The authors wish to express their thanks to prof. K. Aoki and Prof. T. Kaneko for their many valuable suggestions and several discussions.

2. In the following discussions, all manifolds are considered as connected, orientable C^∞ manifolds. By immersion $f: M^n\to R^p$ we mean C^∞ map whose Jacobian matrix has rank $n=\dim M^n$ at each point of M. A homeomorphic immersion will be called imbedding. A manifold M^n will be called parallelizable if its tangent bundle is trivial, we say M^n is almost parallelizable if M^n-x is parallelizable for some $x\in M^n$. M^n will be called π-manifold if M^n is imbedded in R^{n+k} $(k>n)$ with trivial normal bundle ν.

Since, a non-closed (i.e. non-compact or with boundary) almost parallelizable manifold is parallelizable, hence it is immersible in R^{n+1} (Theorem 6.3 of [2]). Therefore we may consider only closed manifolds.

Let o_n denote the obstruction to the extension over M^n of the cross section over M^n-x; o_n is an element of $\pi_{n-1}(SO(k))$. Now let

$$J: \pi_{n-1}(SO(k))\to\pi_{n+k-1}(S^k)$$

be the Hopf-Whitehead homomorphism, it is well known that $J(o_n)=0$. Moreover the result of J. F. Adams [6] implies that homomorphism J is injective for $\pi_{n-1}(SO(k))=Z_2$ $(k>n)$.

In the case of $n\not\equiv 0$ (mod 4), $\pi_{n-1}(SO(k))=0$ or Z_2 according to whether $n\equiv 3,5,6,7$ (mod 8) or $n\equiv 1,2$ (mod 8) respectively. From this it follows that in the case of $n\equiv 3,5,6,7$ (mod 8), $o_n=0$. In the case $n\equiv 1,2$ (mod 8), o_n is also zero, since it belongs to the kernel

*) Details will appear in [7].

of the Hopf-Whitehead homomorphism, which is injective. Hence M^n is a π-manifold, so that it is immersible in R^{n+1},**) this complete the proof of Proposition 1.

In the case $n \equiv 0 \pmod 4$, there exists an almost parallelizable $4k$-dimensional manifold M_o^{4k} with Pontryagin class $p_k[M_o^{4k}] \neq 0$ [5]. By Whitney duality, the $4k$-dimensional Pontryagin class $p_k(\nu)$ of the normal bundle ν coincides with $4k$-dimensional Pontryagin class of M_o^{4k}, up to sign, therefore $p_k(\nu)\ [M_o^{4k}] \neq 0$. Hence it follows from Lemma 1.1 of [3], that $o_n \neq 0$. This means M_o^{4k} is not a τ-manifold, so that it is not immersible in R^{n+1}.

In particular we can prove that M_o^4 is not immersible in R^6, for suppose M_o^4 be immersed in R^6, then the normal bundle ν induced by this immersion is trivial since the second Whitney class $w_2(\nu)=0$, hence by Lemma 6.4 in [2], M_o^4 is immersible in R^5, which is a contradiction. In other words, Hirsch's result is best possible for $n=4$.

References

[1] M. W. Hirsch: Immersions of almost parallelizable manifolds, Proc. Amer. Math. Soc., **12**, 845–846 (1961).

[2] ——: Immersions of manifolds, Trans. Amer. Math. Soc., **93**, 242–276 (1959).

[3] M. Kervaire: On the Pontryagin class of certain $SO(n)$-bundle, Amer. J. Math., **80**, 632–638 (1958).

[4] ——: Sur l'invariant de Smale d'un prongement, Comm. Math. Helv., **34**, 127–139 (1960).

[5] J. Milnor and M. Kervaire: Bernoulli numbers, homotopy groups and a theorem of Rohlin, Proc. International Congress of Math., 454–458 (1958).

[6] J. F. Adams: Vector fields on spheres, Ann. Math., **75**, 603–632 (1962).

[7] T. Watabe and T. Hirose: On the immersibility of almost parallelizable manifolds, to appear in J. Fac. Sci. Niigata Univ., Ser. I, **4**(1).

**) Let M^n be an orientable manifold, then it is necessary and sufficient for M^n to be immersible in R^{n+1} is that it is a π-manifold.