156. On Bochner Transforms. II

A Generalization Attached to M(n, R) and "an" n-Dimensional Bessel Function

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1. **Introduction.** The concept of Bochner transforms is a generalization of Fourier transforms of radial functions. (Bochner [1] and Iwasaki [2].) In this paper we shall define Bochner transforms attached to the space of matrices $M(n, \boldsymbol{R})$ and investigate some of its properties. As an analogy to the case of the one-dimensional Euclidean space we get "n-dimensional Bessel functions". We shall give Bessel differential equations for these functions.

Probably the Bochner transforms have a close relation to Siegel modular functions. We shall discuss in this direction elsewhere.

2. **Definitions and notations.** We denote by $\boldsymbol{P}_0=\boldsymbol{P}_0(n, \boldsymbol{R})$ the space of non-negative symmetric matrices of degree n, by $\boldsymbol{P}$ the set of strictly positive elements in $\boldsymbol{P}_0$ and M_k the space of continuous functions on $\boldsymbol{P}_0$ which is C^∞ on $\boldsymbol{P}$, invariant by the automorphism of $\boldsymbol{P}_0$, $x\to {}^t uxu$ where $u\in U=\boldsymbol{O}(n, \boldsymbol{R})$, and $\int_{\boldsymbol{P}}(\det x)^{\frac{k}{2}}|\varphi(x)|^2dx$ is convergent. Now

Definition. *The Bochner transform* $T=T^n_{\lambda,k}$ *is a linear operator on* M_k *which satisfies the following conditions* (B):

(B$_1$) *the function* $\varepsilon(x)=\exp\left(-\frac{2\pi}{\lambda}\operatorname{tr} x\right)$ *is mapped to itself by* T,

(B$_2$) $\int_U \varphi({}^t w {}^t uxuw)du$ *with* $\varphi\in M_k$ *and* $w\in GL(n, \boldsymbol{R})$ *is mapped by* T, *as a function of* x, *to*

$$\int_U T\varphi(w^{-1}\,{}^t uxu {}^t w^{-1})du\cdot|\det w|^{-k} \quad \left(\begin{matrix}du \text{ is the Haar measure on } U\\ \text{normalized by } \int_U du=1\end{matrix}\right),$$

(B$_3$) $$\int_{\boldsymbol{P}}(\det x)^{\frac{k}{2}}\varphi(x)\psi(x)dx=\int_{\boldsymbol{P}}(\det x)^{\frac{k}{2}}T\varphi(x)T\psi(x)dx,$$

where $\varphi, \psi\in M_k$ *and* dx *is a measure on* $\boldsymbol{P}$ *invariant by* $x\to {}^t wxw$ (see [2]).

Any element φ of M_k is a spherical function on $\boldsymbol{P}$, therefore it has the Fourier transform in Gelfand-Selberg sense. On our stand point it may be called the *Mellin transform* of φ and it is defined as follows (Selberg [3] pp. 56–59):

If x belongs to $\boldsymbol{P}$ it can be represented uniquely as ${}^t tt$, where t is a member of the space of upper trigonal matrices with positive diagonal elements $t_i = t_{ii}$. And

$$\varphi(s)=\varphi(s_1, s_2,\cdots, s_n)=\int_T \varphi({}^t tt) t_1^{s_1} t_2^{s_2+1}\cdots t_n^{s_n+n-1} dt$$
$$=\prod_{i=1}^n \int_0^\infty dt_i \prod_{i<j}\int_{-\infty}^\infty dt_{ij} \varphi({}^t tt)\cdot t_1^{s_1-1}\cdots t_n^{s_n-1}$$

is the Mellin transform of $\varphi(x)$.

By the general theory of spherical functions we know that the ring $\boldsymbol{D}$ of invariant differential operators attached to $\boldsymbol{P}$ has the generators $\Delta_1,\cdots,\Delta_n$ such that

$$\Delta_i \varphi(s)=\sigma_i(s)\varphi(s)$$

where $\sigma_i(s)$ is the fundamental symmetric polynomial in $s_1,\cdots, s_n$ of degree i.

3. Properties. We shall transform the conditions (B) in formulas in s.

By (B_2) and (B_3) we have

$$\int_P (\det x)^{\frac{k}{2}} \varphi(x) \psi({}^t wxw) dx$$
$$=|\det w|^{-k}\int_P (\det x)^{\frac{k}{2}} T\varphi(x) T\psi(w^{-1}x{}^t w^{-1}) dx.$$

Calculating the Mellin tranform (strictly speaking the convolution with the zonal spherical function $\omega_s(w)$) of the both sides as function in w, we get

$$\varphi(k-s)\psi(s)=T\varphi(s)T\psi(k-s),$$

where $k-s$ means $(k-s_1,\cdots, k-s_n)$.

If we take $\varepsilon(x)$ in (B_1) as $\psi(x)$ and use the condition (B_1), the following equality holds:

$$(1) \qquad T\varphi(s)=\left(\frac{2\pi}{\lambda}\right)^{\frac{nk}{2}-(s_1+\cdots+s_n)} \frac{\Gamma\left(\frac{s_1}{2}\right)\cdots\Gamma\left(\frac{s_n}{2}\right)}{\Gamma\left(\frac{k-s_1}{2}\right)\cdots\Gamma\left(\frac{k-s_n}{2}\right)} \varphi(k-s).$$

Proposition 1. $T\varphi(x)$ is equal to

$$\int_P J({}^t tyt)\varphi(y)(\det y)^{\frac{k}{2}} dy$$

where ${}^t tt=x$ and $J(x)=J^n_{\lambda,k}(x)$ is the function whose Mellin transform is

$$\left(\frac{2\pi}{\lambda}\right)^{\frac{nk}{2}-(s_1+\cdots+s_n)} \frac{\Gamma\left(\frac{s_1}{2}\right)\cdots\Gamma\left(\frac{s_n}{2}\right)}{\Gamma\left(\frac{k-s_1}{2}\right)\cdots\Gamma\left(\frac{k-s_n}{2}\right)}.$$

Corollary. $T\varphi$ belongs to M_k and the operator T is continuous with respect to the norm

$$||\varphi||=\left(\int_P (\det x)^{\frac{k}{2}}|\varphi(x)|^2 dx\right)^{\frac{1}{2}}.$$

Proposition 2. T^2 is the identical mapping.

Proposition 3. $\Delta_n T_k\varphi(x)=\left(\frac{4\pi}{\lambda}\right)^n(\det x)T_{k+2}\varphi(x)$.

Proof. For

$$\Delta_n T_k\varphi(s)=\left(\frac{2\pi}{\lambda}\right)^{\frac{nk}{2}-(s_1+\cdots+s_n)}\cdot 2^n\cdot\frac{\Gamma\left(\frac{s_1+2}{2}\right)\cdots\Gamma\left(\frac{s_n+2}{2}\right)}{\Gamma\left(\frac{k-s_1}{2}\right)\cdots\Gamma\left(\frac{k-s_n}{2}\right)}\varphi(k-s).$$

Proposition 4. If $\varphi(x)\in M_k$ is a common eigenfunction for $T^n_{\lambda,k}$ with arbitrary $k>0$, then $\varphi(x)$ is of the form $c\exp\left(-\frac{2\pi}{\lambda}\operatorname{tr} x\right)$.

Proof. By Proposition 2 we have

$$\varphi(s)=\pm T_k\varphi(s) \text{ for } k>0,$$

and by the equality (1) $\left(\frac{2\pi}{\lambda}\right)^{\frac{(k-s_1)+\cdots+(k-s_n)}{2}}\cdot\frac{\varphi(k-s)}{\Gamma\left(\frac{k-s_1}{2}\right)\cdots\Gamma\left(\frac{k-s_n}{2}\right)}$

must be independent of k. Therefore

$$\varphi(s)=c\left(\frac{2\pi}{\lambda}\right)^{-\frac{s_1+\cdots+s_n}{2}}\Gamma\left(\frac{s_1}{2}\right)\cdots\Gamma\left(\frac{s_n}{2}\right).$$

Proposition 5. If $k>0$ the operator defined by the formula (1) is a continuous linear mapping on M_k with respect to the norm

$$||\varphi||^2=\int_P \det x^{\frac{k}{2}}|\varphi(x)|^2 dx.$$

4. Higher dimensional Bessel functions. In the case $n=1$ the function $J^1_{\lambda,k}(t^2)$ is equal to $\left(\frac{4\pi}{\lambda}\right)t^{1-\frac{k}{2}}J_{\frac{k}{2}-1}\left(\frac{4\pi}{\lambda}t\right)$. As an analogy to this case we define the Bessel function of dimension n by the equality

$$\left(\frac{4\pi}{\lambda}\right)(\det x)^{\frac{-\nu}{2}}J^n_\nu\left(\left(\frac{4\pi}{\lambda}\right)^2 x\right)=J^n_{\lambda,2\nu+2}(x)$$

for $x\in \boldsymbol{P}_0$. (Note that $J_\nu(x)=J^1_\nu(x^2)$!) Then we have

$$\left(\frac{4\pi}{\lambda}\right)^{n-(s_1+\cdots+s_n-n\nu)}J^n_\nu(s-\nu)$$
$$=\left(\frac{4\pi}{\lambda}\right)^{\frac{n(2\nu+2)}{2}-(s_1+\cdots+s_n)}\frac{\Gamma\left(\frac{s_1}{2}\right)\cdots\Gamma\left(\frac{s_n}{2}\right)}{\Gamma\left(\nu+1-\frac{s_1}{2}\right)\cdots\Gamma\left(\nu+1-\frac{s_n}{2}\right)}.$$

Proposition 6. $$J^n_\nu(s)=\frac{\Gamma\left(\frac{s_1+\nu}{2}\right)\cdots\Gamma\left(\frac{s_n+\nu}{2}\right)}{\Gamma\left(\frac{\nu+2-s_1}{2}\right)\cdots\Gamma\left(\frac{\nu+2-s_n}{2}\right)}.$$

We have now a few formulas similar to the ordinary Bessel functions.

Proposition 7.

a) $\Delta_n((\det x)^{-\frac{\nu}{2}} J_\nu^n(x)) = 2^n (\det x)^{-\frac{\nu-1}{2}} J_{\nu+1}^n(x)$,

b) $\Delta_n((\det x)^{\frac{\nu-1}{2}} J_\nu^n(x)) = (-2)^n \det x^{\frac{\nu+1}{2}} J_{\nu-1}^n(x)$.

Proof of a). The Mellin transform of both sides are

$$s_1 \cdots s_n \frac{\Gamma\left(\frac{s_1}{2}\right)\cdots}{\Gamma\left(\frac{2\nu+2-s_1}{2}\right)\cdots} \quad \text{and} \quad 2^n \cdot \frac{\Gamma\left(\frac{s_1+2}{2}\right)\cdots}{\Gamma\left(\frac{2\nu+2-s_1}{2}\right)\cdots}.$$

Moreover we have the following Bessel differential equation:

Proposition 8.

$$(\Delta_n - \nu\Delta_{n-1} + \nu^2\Delta_{n-2} - \cdots + (-\nu)^{n-1}\Delta_1 + (-\nu)^n)$$
$$(\Delta_n + \nu\Delta_{n-1} + \cdots + \nu^{n-1}\Delta_1 + \nu^n) J_\nu^n = (-4)^n \det x \cdot J_\nu^n.$$

Proof. $(s_1-\nu)\cdots(s_n-\nu)(s_1+\nu)\cdots(s_n+\nu) J_\nu^n(s) = (-4)^n J_\nu^n(s+2)$.

References

[1] S. Bochner and K. Chandrasekharan: Fourier Transforms, Princeton (1949).
[2] K. Iwasaki: On Bochner transforms, Proc. Japan Acad., **39**(5), 257–262 (1963).
[3] A. Selberg: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., **20**, 47–87 (1956).