# 5. On the Ambiguity of Cut-off Process in the Theory of Quantum Field

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1. In the calculations of quantum field theory, so called "cut-off" procedure is frequently used to remove divergence of some physical quantity. In this and the following [1] notes we discuss the ambiguities which is immanent in some of these cut-off procedures and consider some kind of remedies for it.

2. To clarify the cut-off procedures we consider the relation between the occupation number representation and the tensor product of Hilbert spaces in an axiomatic manner.

Assumptions with respect to the states of a single particle are the following:

$\boldsymbol{S}_1$. The states of a single particle correspond to the vectors of a Hilbert space $\boldsymbol{H}_s$.

$\boldsymbol{S}_2$. The physical quantities of a single particle correspond to the self-adjoint operators (defined on a dense subset of the space $\boldsymbol{H}_s$).

$\boldsymbol{S}_3$. There is a set of physical quantities whose corresponding operators commute each other and makes a complete system of operators. We denote these operators by $O_1,\cdots,O_n$.

$\boldsymbol{S}_4$. We restrict our considerations to the case that eigenvalues $\alpha_i(\alpha_{i1},\cdots,\alpha_{in})$ of the operators $(O_1,\cdots,O_n)$ makes a point (discrete) spectrum.

Assumptions with respect to the states of quantized field are the following:

$\boldsymbol{F}_1$. To every system of eigenvalues $\alpha_i(\alpha_{i1},\cdots,\alpha_{in})$ of single particle, i.e. to every eigenstate $\Psi_i$ of the quantum number $\alpha_i$, one assigns the number $n_i$ of particles which are in this state. We assume that there exist eigenvectors $\psi_{n_i}$ for every $n_i$ and a Hilbert space $\boldsymbol{H}_i$ which is the closure of the linear aggregate of $\{\psi_{n_i}|n_i=0,1,2\cdots\}$.

$\boldsymbol{F}_2$. Corresponding to a state $\Phi$ of the field, there exist vectors of the infinite direct product $\Pi\otimes\boldsymbol{H}_i$ of the space $\boldsymbol{H}_i$ in the sense of J. Von Neumann [2]. We leave details of the indicate direct product to the original author's article [2], and indicate only the following properties (1), (2) of $c$-sequences*) which has close relation to the am-

*) Notations and abbreviations in this article follow J. Von Neumann [2].

biguities of cut-off procedures.

(1) $$\lambda_0 \Pi_{i=1}^{\infty} \otimes f_i = (\Pi_{i=1}^{m-1} \otimes f_i) \otimes \lambda_0 f_m \otimes (\Pi_{i=m+1}^{\infty} \otimes f_i)$$
for any integer $m$ and for any complex number $\lambda_0$,

(2) $$(\Pi_{i=1}^{m-1} \otimes f_i) \otimes (g_m + h_m) \otimes (\Pi_{i=m+1}^{\infty} \otimes f_i)$$
$$= (\Pi_{i=1}^{m-1} \otimes f_i) \otimes g_m \otimes (\Pi_{i=m+1}^{\infty} \otimes f_i) + (\Pi_{i=1}^{m-1} \otimes f_i) \otimes h_m \otimes (\Pi_{i=m+1}^{\infty} \otimes f_i)$$
for any integer $m$.

The vectors having the expressions of either side of these equalities hence define the same state.

3. We consider here some of usual cut-off process in the terms of tensor product. The cut-off procedure which appears in the customary calculation of quantum field theory is of the following kind.

With respect to some physical quantity $\mathfrak{M}$ which is expressed by the function of the divergent integral e.g. $\int_0^{\infty} g(\alpha) d\alpha$, the domain of the integral is cut off as the following: $\int_0^M g(\alpha) d\alpha$ or $\int_\delta^{\infty} g(\alpha) d\alpha$ or $\int_\delta^M g(\alpha) d\alpha$ etc. After these cut-off procedures the physical quantity $\mathfrak{M}\left(\int_0^{\infty} g(\alpha)\, d\alpha\right)$ is frequently calculated as the $\lim_{M\to\infty} \mathfrak{M}\left(\int_0^M g(\alpha)\, d\alpha\right)$ etc. To express these procedures using the corresponding states of field, one defines cut-off operator $P_N$ which maps a vector $\Pi_{i=1}^{\infty} \otimes f_i$ to the vector $(\Pi_{i=1}^{N} \otimes f_i) \otimes (\Pi_{i=N+1}^{\infty} \otimes e_{i0})$ where $e_{i0}$ is a vector which corresponds to the occupation number e.g. 0 for the quantum number $\alpha_i$.

In some cases we may be able to consider that the cut-off proceeds as follows e.g.
$$\lim_{M\to\infty} \int_0^M g(\alpha) d\alpha = \lim_{N\to\infty} \langle \Phi | G | P_N \Psi \rangle \equiv \langle \Phi | G | \Psi \rangle = \int_0^{\infty} g(\alpha) d\alpha.$$

4. Now we consider the ambiguity of these cut-off procedures by the following four examples.

**Example 1.** For any linear aggregate of $c$-sequences $\Phi$ there exists a set of vectors $\Phi^n$ of $\boldsymbol{H}$ such that $P_n \Phi^n$ converges to 0.

**Proof.** For $\Phi = \Sigma_{\nu=1}^{m} \Pi_{\alpha=1}^{\infty} \otimes f_{\alpha\nu} = \Sigma_\nu (\Pi_{\alpha \le N} \otimes \varepsilon_\alpha f_{\alpha\nu} \otimes \Pi_{\beta > N} \delta_\beta f_{\beta\nu})$, we can take $\varepsilon_\alpha, \delta_\beta$ such that the equalities $(\Pi \otimes \varepsilon_\alpha) \otimes (\Pi \otimes \delta_\beta) = 1$ and $(\Pi_{\alpha \le N} \otimes \varepsilon_\alpha(N) f_\alpha) \downarrow 0$ hold. Let $\Phi^N = \Sigma_\nu (\Pi_{\alpha \le N} \otimes \varepsilon_\alpha f_{\alpha\nu} \otimes \Pi_{\beta > N} \delta_\beta f_{\beta\nu})$, then $P_N \Phi^N = \Sigma_\nu (\Pi_{\alpha \le N} \otimes \varepsilon_\alpha f_{\alpha\nu} \otimes \Pi_{\beta > N} \otimes e_{\beta 0})$ converges to 0, since $\|P_N \Phi^N\| \le \Sigma_\nu \|\Pi_{\alpha \le N} \otimes \varepsilon_\alpha f_{\alpha\nu}\| < \varepsilon$. This sort of ambiguity causes form the property (1).

To avoid this sort of ambiguity one may immediately think about the standard forms: $(\Pi_\alpha \|f_\alpha\|) \cdot \Pi_\alpha \otimes (f_\alpha / \|f_\alpha\|)$ for $\Phi = \Pi \otimes f_\alpha$ and $\Sigma_\nu \{(\Pi_\alpha \|f_\alpha\|) \cdot \Pi_\alpha \otimes (f_{\alpha\nu} / \|f_{\alpha\nu}\|)\}$ for $\Phi = \Sigma_\nu \Pi_\alpha \otimes f_{\alpha\nu}$, where $\alpha = 1, 2, 3, \cdots$ and $\nu = 1, 2, \cdots, n$.

One can see however even these standard forms do not always give unique cut-off vector because of the property (2) as the following Example 2 shows.

**Example 2.** Assume that $\varphi=\Pi\otimes f_\alpha+\Pi\otimes g_\alpha$ satisfies the following two conditions: (1) $||f_\alpha||=||g_\alpha||=1$ (2) $\langle f_\alpha, g_\alpha\rangle=0$ for any $\alpha$. Then we see that $\Pi\otimes f_\alpha=\Psi_N\otimes\Psi_\infty$, $\Pi\otimes g_\alpha=\Phi_N\otimes\Phi_\infty$ and $\Psi_N\perp\Phi_N$, $\Psi_\infty\perp\Phi_\infty$, $||\Phi_N||=||\Phi_\infty||=||\Psi_N||=||\Psi_\infty||=1$.

Now

$$\varphi=\Psi_N\otimes\Psi_\infty+\Phi_N\otimes\Phi_\infty=\Psi_N\otimes(\Psi_\infty-\lambda\Phi_\infty)+(\lambda\Psi_N+\Phi_N)\otimes\Phi_\infty.$$

Hence using the first expression,

$$P_N\varphi=\Psi_N\otimes e_{\infty 0}+\Phi_N\otimes e_{\infty 0}=(\Psi_N+\Phi_N)\otimes e_{\infty 0}.$$

So $||P_N\varphi||=\sqrt{2}$. Using the 2nd expression, however,

$$\varphi=||\psi_\infty-\lambda\Phi_\infty||\cdot\Psi_N\otimes\frac{(\Psi_\infty-\lambda\Phi_\infty)}{||\Psi_\infty-\lambda\Phi_\infty||}+||\lambda\Psi_N+\Phi_N||\cdot\frac{(\lambda\psi_N+\Phi_N)}{||\lambda\psi_N+\Phi_N||}\otimes\Phi_\infty.$$

Hence

$$P_N\varphi=\{(||\Psi_\infty-\lambda\Phi_\infty||+\lambda)\Psi_N+\Phi_N\}\otimes e_{\infty 0}.$$

Hence

$||P_N\varphi||=\sqrt{(\sqrt{1+\lambda^2}+\lambda)^2+1}=\sqrt{2}\sqrt{1+2\lambda\sqrt{1+\lambda^2}+\lambda^2}$. So $||P_N\varphi||$ runs through the value from $\sqrt{2}$ to $\infty$. We can see also here

$$\lim_{\lambda\to-\infty}P_N\varphi=\Psi_N\otimes e_{\infty 0} \text{ and } \lim_{\lambda\to+\infty}P_N\varphi=\Psi_N\otimes e_{\infty 0}.$$

The following two examples show more pathological ambiguity of cut-off operators.

**Example 3.** For any state $\Phi$ whose corresponding vector is a $c$-sequence, there exist standard expressions $\Phi^N$ such that $P_N\Phi^N$ converges to 0.

**Example 4.** For any $\Phi, \Psi$ which are expressed by finite linear aggregate of $c$-sequence, there exist sets of expressions $\{\Phi^N\}$ and $\{\Psi^N\}$ such that $P_N\Phi^N=P_N\Psi^N$.

**Proof.** Let $c$-sequence $\Phi=\Pi\otimes f_\alpha=(\Pi_{i=1}^N\otimes f_i)\otimes(\Pi_{i=N+1}^\infty f_i)=\Phi_N\otimes\Phi_\infty$ be a standard form. Let $e_{\infty 1}, e_{\infty 2}$ be a unit vector such that $e_{\infty 1}\perp e_{\infty 2}$ and $||e_{\infty 1}||=||e_{\infty 2}||=1$ and $\Phi_\infty=\frac{1}{\sqrt{2}}(e_{\infty 1}-e_{\infty 2})$. Then we see that $\Phi=\Phi^N$ $=\Phi_N\otimes\left(\frac{1}{\sqrt{2}}e_{\infty 1}-\frac{1}{\sqrt{2}}e_{\infty 2}\right)=\frac{1}{\sqrt{2}}(\Phi_N\otimes e_{\infty 1})-\frac{1}{\sqrt{2}}(\Phi_N\otimes e_{\infty 2})$.

Hence

$$P_N\Phi=\frac{1}{\sqrt{2}}(\Phi_N\otimes e_{\infty 0})-\frac{1}{\sqrt{2}}(\Phi_N\otimes e_{\infty 0})=0.$$

In case $\Phi=\Sigma_{\nu=1}^m\Pi_{\alpha=1}^\infty\otimes f_{\alpha\nu}$, we can prove similarly.

Example 4 is also easily constructed utilizing Example 3.

Lastly it is remarkable that the cut-off process can not always be done in the space $\boldsymbol{H}$ as the following example shows:

**Example 5.** Let $\Phi=\Sigma_{i=1}^\infty c_i\Pi_{n=1}^{i-1}\otimes\varphi_{n1}\otimes\varphi_{i2}\otimes\Pi_{n=i+1}^\infty\otimes\varphi_{n1}$ where $c_i>0$, $\Sigma_{i=1}^\infty c_i^2<\infty$ and $\Sigma_{i=1}^\infty c_i=\infty$ are satisfied. Then we can see easily that $||\Phi||<\infty$ and $||P_N\Phi||=\infty$.

**5.** In view of these ambiguities one may ask the following questions: Under what expressions the customary cut-off procedure is done?

Some answers for these questions will be given in the following articles [1] [3] which also contain some sort of remedies against these ambiguities.

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## References

[1] T. Ishihara: On some topologies in the universal Hilbert space. Proc. Japan Acad., **40**, 23-28 (1964).

[2] J. Von Neumann: On infinite direct product. Comp. Math., **46**, 1-77 (1939).

[3] T. Ishihara: On the universal Hilbert space in the quantum field theory. III (unpublished).