# 1. On Bochner Transforms. III

## Case of $\mathfrak{p}$-adic Number Fields

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1. In the following we shall consider *Bochner transforms* attached to matrices algebras over $\mathfrak{p}$-adic number fields.

Let $k$ be a completion of a finite algebraic number field with respect to a finite prime ideal $\mathfrak{p}$, $\mathfrak{o}$ the ring of integers in $k$, $\pi$ a prime element of $k$ and $\mathfrak{u}$ the unit group. We denote by $A, O, G$, and $U$ the matrices algebra $M(n, k)$, the order $M(n, \mathfrak{o})$, the group $GL(n, k)$ and the unit group of $O$ respectively. Let $\mathcal{F}$ mean the space of the all $U$-biinvariant continuous functions integrable on $A$.

**Definition.** *The Bochner transform $T=T_k^n$ is a linear operator on $\mathcal{F}$ which satisfies the following conditions* (B):

($B_1'$) *the characteristic function $\varepsilon(x)$ of $O$ is mapped to itself by $T$,*

($B_2$) *as a function of $x$, $\int_U \varphi(xuw)du$ with $\varphi \in \mathcal{F}$ and $w \in G$ is mapped to $\int_U T\varphi(xu^t w^{-1})du \, |\det w|_{\mathfrak{p}}^{-k}$ by $T$* ($du$ is the Haar measure of $U$ normalized by $\int_U du=1$),

($B_4$) *there is a $U$-biinvariant continuous function $\alpha(x)$ on $O$ such that*

$$\int_O \alpha(x)\varphi(x)\,dx=\int_O \alpha(x)\,T\varphi(x)\,dx$$

*for any function $\varphi \in \mathcal{F}$* (see [3]).

**Remarks.** (i) The function $\varepsilon(x)$ in ($B_1'$) corresponds to the function $e^{-\pi x^2}$ in ($B_1$) of [3] as $\mathfrak{p}$-component of the function defined on the adele ring appeared in the proof of the functional equation of Riemann zeta-function in the thesis of Tate [4].

(ii) Condition ($B_4$) is an analogy of the *modular relation* (Bochner [2]). On the stand point of Bochner-Chandrasekharan it may be better to consider integrals on an arbitrary compact set. But we treat only the analogy of ordinary modular forms.

(iii) Using the zonal spherical function

$$\omega(w; s)=\omega(w; s_1, s_2, \cdots, s_n)=\int_U |\prod_{i=1}^n t_i(wu)|_{\mathfrak{p}}^{-s_i+(i-1)}du,$$

where $t_i(x)$ is the $i$-th diagonal element of the upper trigonal part

$t$ of a decomposition $x=ut$ with $u\in U$, we can define the Mellin-transform of any function in $\mathcal{F}$.

2. Now we shall determine the function $\alpha(x)$ and the operator $T$.

If we apply ($B_4$) to the function $\int_U \varphi(xuw)du$, then we have by ($B_2$)

$$\int_O \alpha(x)\varphi(xw)dx=\int_O \alpha(x)T\varphi(x^t w^{-1})|\det w|_{\mathfrak{p}}^{-k}dx. \tag{1}$$

As Mellin-transforms of the both sides of (1) we get

$$\int_O \alpha(x)\omega(x;s)dx\cdot\varphi(s)=\int_O \alpha(x)\omega(x;k-s)dx\cdot T\varphi(k-s) \tag{2}$$

for any $\varphi\in\mathcal{F}$. Therefore we have

$$\frac{\varphi(s)}{\varepsilon(s)}=\frac{T\varphi(k-s)}{\varepsilon(k-s)}. \tag{3}$$

But
$$\varepsilon(s)=\int_O \varepsilon(x)\omega(x;s)dx=\int_{O\cap G}\prod_{i=1}^{n}|t_i(x)|_{\mathfrak{p}}^{-s_i+(i-1)}d^\times x$$
$$=\int_{O\cap T}\prod_{i=1}^{n}|t_i|_{\mathfrak{p}}^{-s_i+(i-1)}d^\times t \quad (\boldsymbol{T}\text{ is the set of upper trigonal matrices})$$
$$=\prod_{i=1}^{n}\int_{\mathfrak{o}}|t_i|_{\mathfrak{p}}^{-s_i-1}d^+t_i=\prod_{i=1}^{n}\frac{1}{1-|\pi|_{\mathfrak{p}}^{-s_i}} \quad \text{for } s_i<0.$$

So $T\varphi(s)=\prod_{i=1}^{n}\frac{1-q^{s_i}}{1-q^{k-s_i}}\cdot\varphi(k-s)$ for $-k<s_1<0$ (where $q=|\pi|_{\mathfrak{p}}^{-1}$). And

$$T\varphi(x)=\int_G \mathcal{J}_k^n(xy)\varphi(y)|\det y|_{\mathfrak{p}}^{k}d^\times y,$$

where
$$\mathcal{J}_k^n(x)=\begin{cases}(q^k-1)^n|\det x|_{\mathfrak{p}}^{-k} & x\in O\\ 0 & x\notin\pi^{-1}O\\ (-1)^m(q^k-1)^{n-m}|\det x|_{\mathfrak{p}}^{-k} & x\in\pi^{-1}O \text{ and } \varpi_{i_1}\cdots\varpi_{i_m}x\in U\end{cases}$$

($\varpi_i$ is the diagonal matrix whose $i$-th element is $\pi$ and the others equal to 1).

Now, we apply the formula (1) to the function $\varepsilon(x)$. We have

$$\int_{Ow^{-1}}\alpha(x)dx=\int_{O^t w}\alpha(x)dx|\det w|_{\mathfrak{p}}^{-k},$$

or

$$\int_{O\tilde{\varpi}_{i_1}^{r_1}\cdots\tilde{\varpi}_{i_n}^{r_n}}\alpha(x)dx=\int_O \alpha(x)dx\cdot q^{-k(r_1+\cdots+r_n)}.$$

Therefore

$$\alpha(x)=(1-q^{-k})^{-n}|\det x|_{\mathfrak{p}}^{k-1}\int_O \alpha(x)dx.$$

**Theorem.** *The function $\alpha(x)$ is equal to $c|\det x|_{\mathfrak{p}}^{k-1}$ and the operator $T=T_k^n$ is given by*

$$T\varphi(x)=\int_G \mathcal{J}_k^n(xy)\varphi(y)|\det y|_{\mathfrak{p}}^{k}d^\times y,$$

*where c is a constant and*

$$\mathcal{J}_k^n(x)=\begin{cases}(q^k-1)^n|\det x|_{\mathfrak{p}}^{-k} & \text{if } x\in O,\\ (-1)^m(q^k-1)^{n-m}|\det x|_{\mathfrak{p}}^{-k} & \text{if } x\in\pi^{-1}O \text{ and } \varpi_{i_1}\cdots\varpi_{i_m}x\in U,\\ 0 & \text{if } x\notin\pi^{-1}O.\end{cases}$$

3. We shall investigate the analogy of Bessel functions.

In our case it is natural to think that $|\det x|^{1-\frac{k}{2}}J^n_{\frac{k}{2}-1}(x)=\mathcal{J}_k^n(x)$. Therefore

**Proposition.** *The Bessel function attached to the algebra $M(n,k)$ is given by*

$$J_\nu^n(x)=\begin{cases}(q^{2\nu+2}-1)^n|\det x|_{\mathfrak{p}}^{3\nu+2} & \text{if } x\in O,\\ (-1)^m(q^{2\nu+2}-1)^{n-m}|\det x|_{\mathfrak{p}}^{3\nu+2} & \text{if } x\in\pi^{-1}O \text{ and } \varpi_{i_1}\cdots\varpi_{i_m}x\in U,\\ 0 & \text{if } x\notin\pi^{-1}O.\end{cases}$$

## References

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[3] ——: On Bochner Transform. II. Proc. Japan Acad., **39**(10), (1963).

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