38. *The Mean Continuous Perron Integral*

By Yôto KUBOTA

Hokkaido Gakugei University

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1. Introduction. H. W. Ellis [2] has introduced the *GM*-integral descriptively whose indefinite integral is mean continuous. The *GM*-integral is an extension of the *CP*-integral defined by J. C. Burkill [1]. The aim of this paper is to define an integral of the Perron type which is equivalent to the *GM*-integral. We call this integral the mean continuous Perron integral or *MP*-integral.

In § 2 we shall define the *MP*-integral and prove its fundamental properties. The equivalence between the *GM*-integral and the *MP*-integral will be considered in § 3. The proof is essentially based on the method used by J. Ridder ([4], pp. 7–8).

2. The mean continuous Perron integral.

Definition 2.1 ([2], p. 114). If $f(x)$ is general Denjoy integrable on $[a, b]$ then we write

$$M(f, a, b)=\frac{1}{b-a}\int_a^b f(t)dt.$$

If $\lim_{h\to 0} M(f, c, c+h)=f(c)$ then $f(x)$ is termed mean continuous or M-continuous at c.

Definition 2.2. A finite function $f(x)$ is said to be $\underline{AC}$ on a set E if to each positive number ε, there exists a number $\delta>0$ such that

$$\Sigma\{f(b_k)-f(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequence of intervals $\{(a_k, b_k)\}$ with end points on E and such that $\Sigma(b_k-a_k)<\delta$. There is a corresponding definition of $\overline{AC}$ on E. If the set E is the sum of a countable number of sets E_k on each of which $f(x)$ is $\underline{AC}$ then $f(x)$ is termed $\underline{ACG}$ on E. Similarly we can define $\overline{ACG}$ on E. If $f(x)$ is both $\underline{ACG}$ and $\overline{ACG}$ on E then we say that $f(x)$ is ACG on E.

Definition 2.3 ([2], p. 115). A finite function $f(x)$ is said to be $(\underline{ACG})$ on E if E is the sum of a countable number of closed sets E_k on each of which $f(x)$ is $\underline{AC}$. If "$\underline{AC}$" is replaced by "$\overline{AC}$", then the corresponding definition of $(\overline{ACG})$ is obtained. If $f(x)$ is both $(\underline{ACG})$ and $(\overline{ACG})$ on E then $f(x)$ is termed (ACG) on E.

Definition 2.4. Let $f(x)$ be defined on an interval $[a, b]$. The function $U(x)$ $[L(x)]$ is called upper [lower] function of $f(x)$ in $[a, b]$ if

(i) $U(a)=0$ $[L(a)=0]$,

(ii) $U(x)$ $[L(x)]$ is M-continuous on $[a, b]$,

(iii) $U(x)$ $[L(x)]$ is $(\underline{ACG})$ $[(\overline{ACG})]$ on $[a, b]$,

(iv) $AD\, U(x) \geqq f(x)$ a.e. $[AD\, L(x) \leqq f(x)$ a.e.$]$.

Definition 2.5. If $f(x)$ has upper and lower functions in $[a, b]$ and $\inf U(b)=\sup L(b)$, then $f(x)$ is termed integrable in the mean continuous Perron sense or MP-integrable on $[a, b]$. The common value of the two bounds is called the definite MP-integral of $f(x)$ on $[a, b]$, and is denoted by $(MP)\int_a^b f(t)dt$.

Lemma 2.1 ([2], p. 116). If $f(x)$ is M-continuous and $(\underline{ACG})$ on $[a, b]$ and if $AD\, f(x) \geqq 0$ almost everywhere on $[a, b]$ then $f(x)$ is non-decreasing on $[a, b]$.

The direct consequence of this lemma is the following theorem.

Theorem 2.1. For any upper function $U(x)$ and any lower function $L(x)$, the function $U(x)-L(x)$ is non-decreasing on $[a, b]$.

Theorem 2.2. If $f(x)$ is MP-integrable on $[a, b]$ then $f(x)$ is also so on $[a, x]$ for $a<x<b$.

Proof. For a given $\varepsilon>0$, we can find upper and lower functions $U(x)$ and $L(x)$ such that

$$0 \leqq U(b)-L(b)<\varepsilon.$$

It follows from Theorem 2.1 that $U(x)-L(x)<\varepsilon$ for $a<x<b$, which proves the theorem.

Definition 2.6. Let $f(x)$ be an MP-integrable function on $[a, b]$. Then we define the indefinite MP-integral of $f(x)$ as

$$F(x)=(MP)\int_a^x f(t)dt.$$

Theorem 2.3. For any upper function $U(x)$ and any lower function $L(x)$, $U(x)-F(x)$ $[F(x)-L(x)]$ is non-decreasing on $[a, b]$.

Proof. Let $a \leqq x_1 < x_2 \leqq b$. Then $U(x)-U(x_1)$ is an upper function of $f(x)$ in $[x_1, x_2]$. Hence

$$U(x_2)-U(x_1) \geqq (MP)\int_{x_1}^{x_2} f(t)dt$$

that is,

$$U(x_2)-U(x_1) \geqq F(x_2)-F(x_1)$$

which proves the theorem.

Theorem 2.4. The indefinite integral $F(x)$ is M-continuous on $[a, b]$.

Proof. For a given n $(n=1, 2, \cdots)$ there exists an upper function $U_n(x)$ such that

$$0 \leqq U_n(x)-F(x)<1/n \quad (a \leqq x \leqq b).$$

Hence $U_n(x)$ converges uniformly to $F(x)$ on $[a, b]$. Since $U_n(x)$ is M-continuous, the limit function $F(x)$ is also M-continuous ([1], p. 319).

Theorem 2.5. The indefinite MP-integral $F(x)$ is approximately differentiable almost everywhere and $AD\,F(x)=f(x)$ a.e.

Proof. For a given $\varepsilon>0$ we can find an upper function $U(x)$ such that

$$U(b)-F(b)<\varepsilon^2.$$

We put $R(x)=U(x)-F(x)$. Then $R(x)$ is non-decreasing by Theorem 2.3, and therefore $R'(x)$ is finite almost everywhere and is L-integrable. Hence

$$(L)\int_a^b R'(t)dt \leqq R(b)-R(a)=U(b)-F(b)<\varepsilon^2.$$

We set

$$A(\varepsilon)=\{x: \underline{AD}\,F(x)<f(x)-\varepsilon\},\quad A=\{x: AD\,U(x)<f(x)\}$$

and

$$M=\{x: -\infty<R'(x)<+\infty\}.$$

Then $|A|=0$ and $|M|=b-a$. If $x\in A(\varepsilon)-A$ then

$$\underline{AD}\,F(x)<f(x)-\varepsilon\leqq AD\,U(x)-\varepsilon.$$

Hence

$$AD\,U(x)-\underline{AD}\,F(x)>\varepsilon.$$

If $x\in M$ then $R'(x)=AD\,U(x)-\underline{AD}\,F(x)$. If we put $B(\varepsilon)=\{x: R'(x)>\varepsilon\}$ then it holds that $x\in(A(\varepsilon)-A)\cdot M$ implies $x\in B(\varepsilon)$. Since

$$\varepsilon|B(\varepsilon)|\leqq(L)\int_{B(\varepsilon)} R'(t)dt\leqq(L)\int_a^b R'(t)dt$$

we obtain

$$|B(\varepsilon)|<\varepsilon.$$

Hence

$$|A(\varepsilon)|<\varepsilon.$$

It follows from the relation

$$\{x: \underline{AD}\,F(x)<f(x)\}=\Sigma\{x: \underline{AD}\,F(x)<f(x)-\varepsilon/2^k\}$$

that

$$|\{x: \underline{AD}\,F(x)<f(x)\}|\leqq\Sigma\varepsilon/2^k=\varepsilon.$$

Hence

$$\underline{AD}\,F(x)\geqq f(x) \quad \text{a.e.}$$

Similarly we obtain

$$\overline{AD}\,F(x)\leqq f(x) \quad \text{a.e.,}$$

and therefore

$$AD\,F(x)=f(x) \quad \text{a.e.}$$

3. The relation between the MP-integral and the GM-integral. Ellis [2] has defined the GM-integral in the Denjoy type as follows:

Definition 3.1. Let $f(x)$ be a function defined in $[a, b]$ and suppose there exists a function $F(x)$ such that

(i) $F(x)$ is M-continuous on $[a, b]$,
(ii) $F(x)$ is (ACG) on $[a, b]$,
(iii) $AD\,F(x)=f(x)$ a.e.,

then $f(x)$ is said to be GM-integrable on $[a, b]$ and write

$$(GM)\int_a^b f(t)dt=F(b)-F(a).$$

Theorem 3.1. The MP-integral is equivalent to the GM-integral.

Proof. Suppose that $f(x)$ is GM-integrable on $[a, b]$. Then there exists a function $F(x)$ which is M-continuous, (ACG) and $AD\,F(x)=f(x)$ a.e. Hence the function $F(x)-F(a)$ is an upper function and at the same time a lower function of $f(x)$ in $[a, b]$. Thus $f(x)$ is MP-integrable on $[a, b]$ and

$$(MP)\int_a^b f(t)dt=F(b)-F(a)=(GM)\int_a^b f(t)dt.$$

Next we shall show that the GM-integral includes the MP-integral. Suppose that $f(x)$ is MP-integrable on $[a, b]$ and that

$$F(x)=(MP)\int_a^x f(t)dt.$$

Then $F(x)$ is M-continuous on $[a, b]$ and $AD\,F(x)=f(x)$ a.e. by Theorems 2.4 and 2.5. We must show that $F(x)$ is (ACG) on $[a, b]$. Since $f(x)$ is MP-integrable, there exists a sequence of upper functions $\{U_k(x)\}$ and a sequence of lower functions $\{L_k(x)\}$ such that

$$(1)\qquad \lim U_k(b)=F(b)=\lim L_k(b).$$

Since $U(x)-F(x)$ and $F(x)-L(x)$ are non-decreasing by Theorem 2.3 it holds that

$$(2)\qquad \lim U_k(x)=F(x)=\lim L_k(x) \quad \text{for } a\leqq x\leqq b.$$

The interval $[a, b]$ is expressible as the sum of a countable number of closed sets E_k such that any U_k is AC on any E_k and at the same time any L_k is AC on any E_k. It is sufficient to prove that $F(x)$ is AC on E_k. For this purpose we shall show that $F(x)$ is both $\underline{AC}$ and $\overline{AC}$ on E_k.

Suppose that $F(x)$ is not $\underline{AC}$ on E_k. Then there exists an $\varepsilon>0$ and a finite sequence of non-overlapping intervals $\{(a_\nu, b_\nu)\}$ with end points on E_k such that for any small δ

$$\Sigma(b_\nu-a_\nu)<\delta$$

but it holds

$$(3)\qquad \Sigma\{F(b_\nu)-F(a_\nu)\}\leqq-\varepsilon.$$

Since we can find a natural number p such that

$$U_p(b)-F(b)\leqq 1/2\cdot\varepsilon,$$

and $U_p(x)-F(x)$ is non-decreasing on $[a, b]$, we have

$$(4)\qquad \begin{aligned} &\Sigma\{U_p(b_\nu)-U_p(a_\nu)\}-\Sigma\{F(b_\nu)-F(a_\nu)\} \\ &=\Sigma[\{U_p(b_\nu)-F(b_\nu)\}-\{U_p(a_\nu)-F(a_\nu)\}] \\ &\leqq U_p(b)-F(b)\leqq 1/2\cdot\varepsilon. \end{aligned}$$

It follows from (3) and (4) that

$$\begin{aligned} \Sigma\{U_p(b_\nu)-U(a_\nu)\} &\leqq \Sigma\{F(b_\nu)-F(a_\nu)\}+1/2\cdot\varepsilon \\ &\leqq -1/2\cdot\varepsilon. \end{aligned}$$

This contradicts the fact that $U_p(x)$ is $\underline{AC}$ on E_k. Hence $F(x)$ is $\underline{AC}$ on E_k.

Similarly we can prove that $F(x)$ is $\overline{AC}$ on E_k. Thus $F(x)$ is (ACG) on $[a, b]$. This completes the proof.

References

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