36. A Property of Green's Star Domain

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Let R be a hyperbolic Riemann surface and $g(p, o)$ be the Green function on R with its pole o in R. The Geen's star domain $R^{g,o}$ with respect to R and o is the set of points in R which can be joined by Green arcs issuing from o. We also assume that o is a member of $R^{g,o}$. We shall see that $R^{g,o}$ is a simply connected domain. Hence we can map $R^{g,o}$ onto the open unit circular disc by a one-to-one conformal mapping φ. We shall show that the image of a singular Green line (i.e. a Green line on which $g(p, o)$ has a positive infimum) issuing from o by φ is a Jordan curve starting from $\varphi(o)$ and terminating at a point of the unit circumference. We denote by N_φ the totality of end points on the unit circumference of image curves of singular Green lines issuing from o by the mapping φ. The main purpose of this paper is to show that N_φ is of logarithmic capacity zero.

1. Let R be a hyperbolic Riemann surface. This means that there exists the Green function $g(p, o)$ with the arbitrary given pole o in R. We define the pair $(r(p), \theta(p))$ of local functions on R by the relations

$$\begin{cases} dr(p)/r(p) = -dg(p, o) \\ d\theta(p) \qquad = -{}^*dg(p, o). \end{cases}$$

By giving the initial condition $r(o)=0$, $r(p)$ is the global function $e^{-g(p,o)}$ on R. Each branch of $r(p)e^{i\theta(p)}$ can be taken as a local parameter at each point of R except possibly a countable number of points at which $d\theta(p)=0$. A Green arc is an open arc on which $\theta(p)$ is a constant, being considered locally, and $d\theta(p)\neq 0$. A *Green line* is a maximal Green arc. We denote by $G(R, o)$ the totality of Green lines issuing from o. We set, for each $L\in G(R, o)$,

$$d(L)=\sup(r(p);\ p\in L).$$

Clearly $0<d(L)\leq 1$. We say that $L(\in G(R, o))$ is a *singular Green line* if $d(L)<1$. We denote by $N(R, o)$ the set of all singular Green lines in $G(R, o)$. We also denote by $E(R, o)$ the totality of L in $G(R, o)$ such that the closure of L contains a point $p(\neq o)$ with $d\theta(p)=0$. Clearly $G(R, o)\supset N(R, o)\supset E(R, o)$. We set

$$R^{g,o}=(o)\cup(p\in R;\ p\in L \text{ for some } L \text{ in } G(R, o)).$$

We call the set $R^{g,o}$ the *Green's star domain* with respect to R and o. Then we see that

LEMMA 1. *The Green's star domain $R^{g,o}$ is a simply connected domain.*

Proof. Let q be a point in $R^{g,o}$. Then there exists one and only one L in $G(R, o)$ with $q \in L$. We denote by $L(q)$ the half closed subarc of L joining o and q. Since each branch of $r(p)e^{i\theta(p)}$ is a local parameter at each point of $L(q) \cup (o)$ and $\theta(p)$ is a constant on $L(q)$, we can find a simply connected domain G containing $L(q) \cup (o)$ such that $r(p)e^{i\theta(p)}$ is one-valued and univalent in G. We take a branch ϕ of $re^{i\theta(p)}$ such that $\arg(\phi(p))=0$ on $L(q)$. Then G contains a sectorial domain $S_q=\phi^{-1}(re^{i\theta};\ 0<r<\rho,\ -\varepsilon<\theta<\varepsilon)$ $(\rho>r(q),\ \varepsilon>0)$. Since $\phi^{-1}(re^{i\theta};\ 0<r<\rho,\ \theta=\eta)$ $(-\varepsilon<\eta<\varepsilon)$ is a Green arc issuing from o, S_q is contained in $R^{g,o}$. Hence we see that

(1) for any $L(q)$, there exists a sectorial domain S_q such that

$$L(q) \subset S_q \subset R^{g,o}.$$

From this, we see that $R^{g,o}$ is an open set. As each point q in $R^{g,o}$ can be joined by the arc $L(q) \cup (o)$ with o in $R^{g,o}$, so $R^{g,o}$ is connected. Hence $R^{g,o}$ is a domain. Next we show that $R^{g,o}$ is simply connected. Let J be a closed Jordan curve in $R^{g,o}$ and $p=p(t)$ $(0 \leq t \leq 1)$ is a continuous representation of J. For each $\tau (0 \leq \tau \leq 1)$, we denote by $p(t, \tau)$ the point on $L(p(t)) \cup (o)$ such that $r(p(t, \tau)) : r(p(t)) = 1-\tau : 1$. Then from (1), it is easy to see that $J_\tau : p=p(t, \tau)$ $(0 \leq t \leq 1)$ is a closed Jordan curve in $R^{g,o}$. Moreover, by using (1), $p(t, \tau)$ is seen to be a continuous mapping of $(0 \leq t \leq 1) \times (0 \leq \tau \leq 1)$ into $R^{g,o}$. Since $p(t, 0)=p(t)$ and $p(t, 1)=o$, J_τ $(0 \leq \tau \leq 1)$ is a continuous deformation of J to the one point o in $R^{g,o}$. Thus $R^{g,o}$ is simply connected. Q.E.D.

LEMMA 2. *Let φ be a one-to-one conformal mapping of $R^{g,o}$ onto the open unit circular disc $U: |z|<1$ and $L_\varphi=\varphi(L)$ for L in $G(R, o)$. Then L_φ is a Jordan arc in U starting from $\varphi(o)$ and terminating at a point of the unit circumference $C: |z|=1$.*

Proof. Let ϕ be a branch of $r(p)e^{i\theta(p)}$ on $R^{g,o}$. Then ϕ is a one-valued analytic function in $R^{g,o}$ and

$$\alpha_L = \lim_{p \in L, r(p) \nearrow d(L)} \phi(p)$$

exists for any L in $G(R, o)$. It is clear that L_φ is a Jordan arc in U and $\bar{L} \frown C \neq \phi$. Contrary to the assertion, assume that $\bar{L} \frown C$ is not one point. Then applying Theorem of Koebe-Gross (see p. 5 in Noshiro's book [3]) to the function $\phi(\varphi^{-1}(z))$, we conclude that $\phi(\varphi^{-1}(z))$ is identically α_L in U, which is a contradiction. Q.E.D.

2. As before, let φ be a one-to-one conformal mapping of $R^{g,o}$ onto $U: |z|<1$. We denote by z_L the point on $C: |z|=1$ at which $L_\varphi=\varphi(L)$ $(L \in G(R, o))$ terminates. We set

$$N_\varphi = (z_L;\ L \in N(R, o)).$$

Similarly we set $E_\varphi=(z_L;\ L \in E(R, o))$. Clearly E_φ is contained in N_φ. Now we state our main result.

THEOREM 1. *The outer logarithmic capacity of N_φ is zero.*

Proof. Let $(R_n)_{n=1}^{\infty}$ be a normal exhaustion of R with $o\in R_1$. For each positive number a, we set $V(a, n)=(R-R_n)\frown(p\in R;\ g(p, o)\geq a)$. We say that a is admissible if $V(a, n)\neq\phi$ for all positive integers n. Then there exists a positive harmonic function $u(p)$ on R such that

(2) $$\lim_{n\to\infty}\inf_{p\in V(a,n)}u(p)=\infty$$

for any admissible positive number a and

(3) $$D_R(\min(u(p), c))=\iint_R |\mathrm{grad}\ \min(u(p(z)), c)|^2 dxdy\leq 2\pi c$$

for any positive number c, where $u(p(z))$ is a local representation of $u(p)$ by the local parameter $z=x+iy$. For the existence of such a function $u(p)$, see Nakai [2].

Let $v=u\circ\varphi^{-1}$ in $U:|z|<1$ and v^* be a conjugate harmonic function of v in U. Then

$$f(z)=1/(v(z)+iv^*(z))$$

is an analytic function in U with strictly positive real part in U. Let L be in $N(R, o)-E(R, o)$. Then $L\frown(R-R_n)\subset V(-\log d(L), n)$. Hence by (2), $u(p)$ has the asymptotic value ∞ along L. Thus $v(z)$ has the asymptotic value ∞ along L_φ and so $f(z)$ has the asymptotic value 0 along L_φ. Since $\mathrm{Re}[f(z)]>0$ in U, by Theorem of Lindelöf-Iversen-Gross (see p. 5 in Noshiro's book [3]), we get that

(4) $f(z)$ has the angular limit 0 at each point $e^{i\theta}$ in $N_\varphi-E_\varphi$.

Let Φ be the Riemann covering surface of the w-plane generated by $w=f(z)$ and $s(\rho)$ denote the spherical area of the part of Φ above $|w|<\rho$. Put $\Delta(\rho)=(z\in U;\ |f(z)|<\rho)$ and $\Delta_n=\Delta(\rho/2^n)-\Delta(\rho/2^{n+1})$. Then, since

$$|v(z)|\leq 1/|f(z)|\leq 2^{n+1}/\rho$$

on Δ_n, by using (3), we get

$$\begin{aligned}
s(\rho/2^n)-s(\rho/2^{n+1}) &= \iint_{\Delta_n}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}rdrd\theta,\quad z=re^{i\theta}\\
&= \iint_{\Delta_n}\frac{|\mathrm{grad}\ v(z)|^2|f(z)|^4}{(1+|f(z)|^2)^2}rdrd\theta\\
&\leq \iint_{\Delta_n}(\rho/2^n)^4|\mathrm{grad}\ v(z)|^2 rdrd\theta\\
&\leq (\rho^4/2^{4n})D_R(\min(u(p(z)), 2^{n+1}/\rho))\\
&\leq (\rho^4/2^{4n})2\pi(2^{n+1}/\rho)=4\pi(\rho^3/8^n).
\end{aligned}$$

Hence we get that

$$s(\rho)=\Sigma_{n=0}^{\infty}(s(\rho/2^n)-s(\rho/2^{n+1}))\leq 4\pi\Sigma_{n=0}^{\infty}\rho^3/8^n,$$

that is

(5) $$s(\rho)\leq 5\pi\rho^3\quad(\rho>0).$$

On the other hand, we denote by $\alpha(\rho)$ the spherical area of the part of Φ above $|w|\geq\rho$. We put $\nabla(\rho)=(z\in U;\ |f(z)|\geq\rho)$. Since

$$|v(z)|\leq 1/|f(z)|\leq 1/\rho$$

on $V(\rho)$, similarly as above, we get

$$a(\rho)=\iint_{V(\rho)}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}rdrd\theta,\ z=re^{i\theta}$$
$$=\iint_{V(\rho)}\frac{|\operatorname{grad} v(z)|^2|f(z)|^4}{(1+|f(z)|^2)^2}rdrd\theta$$
$$\leq\iint_{V(\rho)}|\operatorname{grad} v(z)|^2 rdrd\theta$$
$$\leq D(\min(u(p),1/\rho)\leq 2\pi/\rho.$$

From this and (5), it follows that

(6) the spherical area of Φ is finite.

It also follows from (5) that

(7) $$\liminf_{\rho\to 0} s(\rho)/\pi\rho^2=0.$$

This means that 0 is an ordinary value of $f(z)$ in the sense of Beurling. Thus from (6) and (7), by using Beurling's theorem (see Theorem 6, p. 114 in Noshiro's book [3]), we conclude that the set X of points in C: $|z|=1$, where $f(z)$ has the angular limit 0, is of logarithmic capacity zero. From (4), it follows that $N_\varphi-E_\varphi\subset X$ and since E_φ is at most countable, we conclude that N_φ is of logarithmic capacity zero. This completes the proof.

3. Take a branch ϕ of $r(p)e^{i\theta(p)}$ on $R^{g,o}$. Then ϕ is a one-to-one conformal mapping of $R^{g,o}$ onto the "radial slits disc"

$$U^{g,o}=(z;\ |z|<1)-\bigcup_{t\in N}(\rho e^{it};\ 0<\varepsilon_t\leq\rho<1),$$

where N is a subset of $(0, 2\pi]$ such that for each $t\in N$, $(\rho e^{it};\ 0<\rho<\varepsilon_t)$ is the image of singular Green line in $G(R, o)$ by ϕ. We also denote by E the totality of $t\in N$ such that $\varepsilon_t e^{it}$ is the image of branch point of $g(p, o)$ in R by ϕ. We can use the function $u(p)$ in the proof of Theorem 1 to prove the following

THEOREM 2 (Brelot-Choquet [1]). *The linear measure of N is zero.*

Proof. Let $N_a(a>0)$ denote the set $(t;\ t\in N,\ \varepsilon_t\leq a)$. Since measure $(N)=\sup_{a>0}$ measure (N_a), we have only to show that measure $(N_a)=0$ for an $a>0$. Clearly $(0, 2\pi]-N_a$ is open and so N_a is closed. Since E is countable, N_a-E is a Borel set. Let $w(z)=u(\phi^{-1}(z))$ on $U^{g,o}$, where u is as in the proof of Theorem 1. Then by (2), for any $t\in N_a-E$,

(8) $$\lim_{\rho\nearrow\varepsilon_t} w(\rho e^{it})=\infty.$$

From (3), it also follows that

(9) $$D_{U^{g,o}}(\min(w(z), c))\leq 2\pi c$$

for any positive number c. From (8), we get

$$c-d\leq\int_\varepsilon^{\varepsilon_t}\frac{\partial}{\partial r}w_c(re^{it})dr,$$

where $t\in N_a-E$ and ε is a small positive number such as $(z;\ |z|\leq\varepsilon)$

$\subset U^{g,o}$ and $d=\sup_{|z|=\varepsilon}w(z)$ and $w_c(z)=\min(w(z),c)$. By Schwarz's inequality, if $c>d$,

$$(c-d)^2\leq\int_{\varepsilon}^{\varepsilon_t}\left|\frac{\partial}{\partial r}w_c(re^{it})\right|^2 rdr\cdot\log\frac{a}{\varepsilon}.$$

Thus we get

$$\begin{aligned}&(c-d)^2 \text{ measure}(N_a)\\ &\leq\iint_{N_a\,\varepsilon}^{\varepsilon_t}\left(\left|\frac{\partial}{\partial r}w_c(re^{it})\right|^2+\frac{1}{r^2}\left|\frac{\partial}{\partial t}w_c(re^{it})\right|^2\right)rdrdt\cdot\log\frac{a}{\varepsilon}\\ &\leq D_{U^{g,o}}(\min(w(z),c)\cdot\log\frac{a}{\varepsilon}\leq 2\pi c\cdot\log\frac{a}{\varepsilon}.\end{aligned}$$

Hence

$$\text{measure}(N_a)\leq 2\pi c(c-d)^{-2}\cdot\log\frac{a}{\varepsilon}.$$

Since c is arbitrary, we conclude that measure$(N_a)=0$ by making $c\nearrow\infty$. Q.E.D.

It may be still an open question whether the logarithmic capacity of N is zero or not. However, our Theorem 1 assures that if we map $U^{g,o}$ onto $U:|z|<1$ one-to-one conformally, then the image of $(\varepsilon_t e^{it};\ t\in N)$ is of logarithmic capacity zero.

References

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[2] M. Nakai: Green potential of Evans type on Royden's compactification of a Riemann surface, to appear.

[3] K. Noshiro: Cluster Sets. Springer (1960).