34. A Characterization of Finite Projective Linear Groups

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We say that a group G admits a Bruhat decomposition if it satisfies the following conditions (1) through (11).[1]

(1) G has three subgroups U, H, and W.

(2) H normalizes U.

(3) H is a normal subgroup of W.

(4) $W/H=\mathfrak{W}$ is a finite group.

(5) For each $w\in\mathfrak{W}$, U has subgroups U'_w and U''_w such that $U=U'_wU''_w$. For each $w\in\mathfrak{W}$, choose a representative $\omega(w)$ in W.

(6) $\omega(w)^{-1}U'_w\omega(w)\subset U$.

(7) $G=\sum_{w\in\mathfrak{W}}HU\omega(w)U''_w$, and in the representation $g=hu\omega(w)u''$ with $h\in H$, $u\in U$, $w\in\mathfrak{W}$ and $u''\in U''_w$, each factor is unique.

(8) There is a distinguished set $\mathfrak{F}$ of elements of $\mathfrak{W}$ of period 2 which generates $\mathfrak{W}$.

(9) For $w\in\mathfrak{F}$ and $s\in\mathfrak{W}$, $U''_w\subseteqq U'_s$ implies $U''_w\subseteqq U'_{sw}$.

(10) For $w\in\mathfrak{F}$, $HU+HU\omega(w)U''_w$ is a subgroup of G.

(11) There is an $x\in U$ such that $x\in U'_w$ implies $w=1$.

The group $\mathfrak{W}$ is called Weyl group associated to this Bruhat decomposition and the set $\mathfrak{F}$ is called the canonical set of generators of $\mathfrak{W}$.

It is well known that the projective special linear group $PSL(n, q)$, operating on the Desarguesian projective space of dimension $n-1$ over Galois field $GF(q)$, admits a Bruhat decomposition with the symmetric group S_n of degree n as Weyl group and with the set $\mathfrak{F}=\{(1, 2), (2, 3), \cdots, (n-1, n)\}$ as the canonical set of generators of $\mathfrak{W}$, where elements of S_n operate on n letters $1, 2, \cdots, n$.

Recently D. G. Higman and J. E. Maclaughlin proved in [2] that a finite group G admitting a Bruhat decomposition with the symmetric group of degree 3 as Weyl group has a representation θ on a finite Desarquesian projective plane such that $\theta(G)$ contains the group $PSL(3, q)$. As a generalization of this theorem, we prove the following

Theorem. *Let a finite group G admit a Bruhat decomposition with the symmetric group S_n of degree $n\geqq 4$ as Weyl group $\mathfrak{W}$ and with the canonical set $\mathfrak{F}=\{(1, 2), \cdots, (n-1, n)\}$ of generators of $\mathfrak{W}$ where elements of S_n operate on n letters $1, \cdots, n$. Then G has a representation θ on a finite projective space such that $\theta(G)$ contains*

1) See R. Steinberg [4].

the group $PSL(n, q)$ or the alternating group A_7 of degree 7. *Hence, if G is simple, it is isomorphic with $PSL(n, q)$ or A_7.*

We want here to sketch our proof of the theorem. A complete proof will be given elsewhere.

Let $G^{(r)}=\sum_{w\in S_r\times S_{n-r}} HU\omega(w)U$ for $r=1,\cdots,n-1$, where S_r is the symmetric group on r letters $1,\cdots,r$ and S_{n-1} is the symmetric group on $n-r$ letters $r+1,\cdots,n$. Let $\mathfrak{S}$ be the set consisting of all cosets G by $G^{(r)}$, $r=1,\cdots,n-1$. If $G^{(r)}a$ and $G^{(s)}b$ are two elements of $\mathfrak{S}$, if $r\geqq s$ and $G^{(r)}a\frown G^{(s)}b\neq\phi$, we say that $G^{(r)}a$ contains $G^{(s)}b$ and we denote by $G^{(r)}a\geqq G^{(s)}b$. Then we can prove that $\mathfrak{S}$ is a semi-ordered set, and indeed, a lattice, and the subset $\mathfrak{P}$ of $\mathfrak{S}$ consisting of all cosets of G by $G^{(1)}$ and $G^{(2)}$ is a projective space in the sense of O. Veblen and J. W. Young, where each coset G by $G^{(1)}$ is a point and each coset of G by $G^{(2)}$ is a line.[2] Moreover we can prove that there exists a one to one correspondence between the set consisting of all r-dimensional subspaces of $\mathfrak{P}$ and the set consisting of all cosets of G by $G^{(r+1)}$, and, by this correspondence, the lattice $\mathfrak{S}'$ consisting of all subspaces of $\mathfrak{P}$ is lattice isomorphic to the lattice $\mathfrak{S}$. If we define operation of elements of G on $\mathfrak{S}$ by $(G^{(r)}a)^x=G(ax)$ where $G^{(r)}a\in\mathfrak{S}$ and $x\in G$, then we can prove that G is flag transitive on $\mathfrak{S}$ in the sense of D. G. Higman.[3] Hence, by the theorem of D. G. Higman, our theorem is proved.[4]

References

[1] R. Baer: Linear Algebra and Projective Geometry. New York (1952).

[2] D. G. Higman and J. E. Mclaughlin: Geometric *ABA*-groups. Illinois Jour. of Math., **5**, 382-397 (1961).

[3] D. G. Higman: Flag-transitive collineation groups of finite projective spaces. Illinois Jour. of Math., **6**, 434-446 (1962).

[4] R. Steinberg: Prime power representations of finite linear groups II. Canadian Jour. of Math., **9**, 347-351 (1957).

[5] O. Veblen and J. W. Young: Projective Geometry I, II. Boston (1909).

2) See O. Veblen and J. W. Young [5].

3) See D. G. Higman [3].

4) See D. G. Higman [3] and R. Baer [1], Chapter VII.