71. *On Some Singular Integral Equations. I*

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1. The theory of linear singular line integral equations with Cauchy-type kernel, on which extensive work has been done, is already a classical one. Beautiful unified results have been published in Muskhelishvili's book [1]; however, work is still being done on this theory and on that of some nonlinear line equations. Muskhelishvili worked on a linear line singular integral equation of the second kind first, reducing it to the Hilbert problem, and solved the "dominant" equation in which the kernel is $1/(t-t_0)$. With regard to a general equation of the second kind, he says only that it is reduced to a Fredholm integral equation by the application of the solution for the dominant equation, i.e., of the inverse operator of the dominant operator, on the general equation of the second kind. Therefore, no compact formulation is obtained by this method for a solution of a general equation. The theory of equations of the first kind, in his method, is included in that of equations of the second kind as a special case, and no compact formulation for a solution is given.

The author encountered an equation of the first kind while working on certain Dirichlet and Neumann problems for the wave equation [2], but he solved it by a revised form of Muskhelishvili's method. The reasons why he was not satisfied with Muskhelishvili's method are as follows: (1) Consideration of an equation of the second kind first results in unnecessary complication of the method, and (2) for the purpose of solving a singular integral equation, it is not necessary to investigate the Hilbert problem as precisely as Muskhelishivili did if an equation of the first kind is solved first. In this paper it will be shown first how to derive a compact formulation for solutions of a linear, singular line equation of the first kind directly, without referring to a Fredholm integral equation, and then how to derive a compact formulation for the solution of an equation of the second kind by reducing it to an equation of the first kind. A general case will be treated, in which the path of integration L is a mixture of closed contours and arcs.

The method will be generalized to the cases of a singular surface integral equation and of some nonlinear integral equations in the

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following paper.

2. Let $L\equiv\sum_{i=1}^{\nu}L_i=\sum_{j=1}^{\nu'}L'_j+\sum_{k=1}^{\nu''}L''_k,\quad(\nu=\nu'+\nu'')$
be a union of ν smooth and non-intersecting closed contours and arcs, where the contours are specifically denoted by L'_j $(j=1,2,\cdots,\nu')$ and arcs by L''_k $(k=1,2,\cdots,\nu'')$. End points of arcs are denoted by C_l $(l=1,2,\cdots,2\nu'')$, which are numbered so that solutions of singular integral equations investigated in the following are bounded at C_l $(l=1,2,\cdots,r)$ and are unbounded at C_l $(l=r+1,r+2,\cdots,2\nu'')$. If the direction of L is given, a neighborhood $U(L''_k)$ of an arc L''_k is separated as $U^+(L''_k)$ and $U^-(L''_k)$ except in the vicinity of C_l, where $U^+(U^-)$ lies on the plus (minus) side of L''_k with respect to its direction. Similarly, when L_i is a closed contour L'_j, then the whole plane is separated as $U^+(L'_j)$ and $U^-(L'_j)$, where $U^+(U)^-$ is the interior (exterior) of L'_j.

In this section, a linear singular line integral equation of the first kind

$$\hat{k}\tau\equiv\frac{1}{\pi i}\int_L^*\left\{\frac{1}{t-t_0}-k(t_0,t)\right\}\tau(t)dt=f(t_0)\tag{1}$$

will be investigated, where t and t_0 are points on L, while $f(t)$, $\tau(t)\in H(L)$, where $H(L)$ is a set of functions which satisfy a Hölder condition on L. The symbol * means that the integral is taken in the sense of Cauchy's principal value. $k(t_0,t)$ is a given function such that $k(t_0,t)=k^*(t_0,t)/|t-t_0|^\alpha$ where $k^*(t_0,t)\in H(L)$ and $0\leqq\alpha<1$. The following definitions are also necessary.

Definition. (i) $X(z)=\prod_{i=1}^{\nu}X_i(z)$, where z is a point in a plane, and $X_i(z)$ is defined as follows:

$$X_i(z)=+1,\text{ in }U^+(L_i),\ -1,\text{ in }U^-(L_i),$$

where L_i is a contour L'_j, and

$$\prod_{i=1}^{\nu''}X_i(z)=\sqrt{\prod_{l=1}^{r}(z-c_l)}\Big/\sqrt{\prod_{l=r+1}^{2\nu''}(z-c_l)}$$

where L_i are arcs L''_k. The right hand side of the last expression is understood to refer to that branch which is holomorphic in the plane cut along L.

(ii) The limiting values of $X(z)$ when $z\to t_0\in L$ are denoted as

$$X^+(t_0)\equiv X(t_0)=\lim_{z\to t_0}X(z),\quad\text{when }z\in U^+(L),$$
$$X^-(t_0)=-X(t_0)=\lim_{z\to t_0}X(z),\quad\text{when }z\in U^-(L).$$

(iii) A singular operator Λ is defined by

$$\Lambda\varphi(t_0)\equiv\frac{X(t_0)}{\pi i}\int_L^*\left\{\frac{1}{t-t_0}-\lambda(t_0,t)\right\}\frac{\varphi(t)}{X(t)}dt$$

where $\lambda(t_0,t)$ is a given function which will be determined later.

(iv) An operator K is defined as

$$K\varphi(t_0)\equiv\int_L K(t_0, t)\varphi(t)dt$$

where $K(t_0, t)$ is given by

$$K(t_0, t)\equiv\frac{1}{\pi^2}\int_L \frac{X(\zeta)}{X(t)}\left\{\frac{1}{\zeta-t_0}-k(t_0, \zeta)\right\}\left\{\frac{1}{t-\zeta}-\lambda(\zeta, t)\right\}d\zeta.$$

Remark 1. It is not difficult to prove that $\lim_{t\to t_0}(t-t_0)K(t_0, t)=0$. Hence K is not a singular operator.

In terms of these notations, we can prove the following theorems.

Theorem 1. For any $g(t)\in H(L)$, we have the identity

$$\hat{k}(\Lambda g+g^{(0)})=(I-K)g \tag{2}$$

where I is the identity operator and $g^{(0)}$ is a solution of $\hat{k}g^{(0)}=0$.

Corollary 1. If $(I-K)^{-1}$ exists, then we have the identity relation

$$\hat{k}\Lambda(I-K)^{-1}\equiv I. \tag{3}$$

Theorem 1 is proved with the help of the Poincaré-Bertrand theorem [1].

Remark 2. Though both Λ and K depend on the choice of $\lambda(t_0, t)$, Corollary 1 shows that $\Lambda(I-K)^{-1}$ is independent of $\lambda(t_0, t)$.

Theorem 2. Since K is not a singular operator (see Remark 1), one has

$$\hat{k}K=K\hat{k},\ \Lambda K=K\Lambda. \tag{4}$$

However, $\hat{k}$ and Λ are not interchangeable in general.

Theorem 3. If $(I-K)^{-1}$ exists, then

$$\hat{k}(I-K)^{-1}=(I-K)^{-1}\hat{k} \tag{5}$$

With the help of these results, one obtains the following fundamental theorems.

Theorem 4.

$$\hat{k}\tau=f \iff (I-K)\tau=\Lambda f+\tau^{(0)}, \tag{6}$$

where

$$\hat{k}\tau^{(0)}=0.$$

Theorem 5. If (6) has a solution, then $(I-K)^{-1}$ must exist and the solution (6) must be given by

$$\tau=(I-K)^{-1}\Lambda f+\tau^{(0)}. \tag{8}$$

Conversely, we can prove that (8) satisfies (6), with the help of theorems mentioned above.

(8) is the desired formulation for the solution of (6). However, we must note one more thing.

Since the operator K, or the kernel $K(t_0, t)$, depends on $\lambda(t_0, t)$, we may choose $\lambda(t_0, t)$ so that

$$||K||<1. \tag{9}$$

In other words, $\lambda(t_0, t)$ should be determined subject to the restriction (9). Then (8) is replaced by

$$\tau=\Lambda\sum_{n=0}^{\infty}f_n+\tau^{(0)} \tag{10}$$

where

$$f_n=Kf_{n-1}=K_nf, \quad f_0=f, \tag{11}$$

and K_n is an operator with iterated kernel,

$$K_n(t_0, t)\equiv\int_L K_{n-r}(t_0, \zeta)K_r(\zeta, t)d\zeta$$

$$K_1(t_0, t)=K(t_0, t), \quad (r=1, 2,\cdots, n-1;\ n=2, 3,\cdots). \tag{12}$$

(10) is a better formulation for the solution to (6), in the sense that it is more precise than (8) and is more convenient for practical calculations.

3. In this section, a singular integral equation of the second kind

$$\hat{K}\tau\equiv A(t_0)\tau(t_0)+B(t_0)\hat{k}\tau(t_0)=f(t_0) \tag{13}$$

will be investigated, where $\hat{k}\tau$ is as was defined in (1), and $A, B, f, \tau \in H(L)$. We assume that $A^2(t)-B^2(t)\neq 0$ on L.

Suppose that $C(t)$ and $D(t)$ are functions such that $C, D\in H(L)$ and

$$A(t)C(t)+B(t)D(t)\equiv 0, \quad A(t)D(t)+B(t)C(t)\equiv 1.$$

Assume that an operator of the second kind Γ and its adjoint operator Γ' are defined as follows:

$$\Gamma\varphi(t_0)\equiv C(t_0)\varphi(t_0)+\frac{D(t_0)}{\pi i}\int_L^*\left\{\frac{1}{t-t_0}-\gamma(t_0, t)\right\}\varphi(t)dt,$$

$$\Gamma'\psi(t_0)\equiv C(t_0)\psi(t_0)-\frac{1}{\pi i}\int_L^*\left\{\frac{1}{t-t_0}+\gamma(t, t_0)\right\}D(t)\psi(t)dt,$$

where $\gamma(t_0, t)$ is a given function which will be determined later. Then, it is not difficult to show that

$$\hat{k}^*\equiv\hat{K}\Gamma \tag{14}$$

is a singular operator of the first kind.

Let us define the inner product for $x, y\in H$, by $(x, y)\equiv\int_L x(t)y(t)dt$, then the following theorem is known [1]:

Theorem 6. The necessary and sufficient conditions for solvability $\Gamma y=x$ are

$$(x, \psi_j)=0, \quad (j=1, 2,\cdots, n)$$

where $\{\psi_j\}$ is an orthonormal system of independent solutions of $\Gamma'\psi=0$.

From this theorem, we can prove

Corollary 6. For any function $x\in H(L)$, the equation

$$\Gamma y=x^*$$

is solvable, where

$$x^*\equiv x-\sum_{j=1}^{n}(x, \psi_j)\psi_j.$$

Now, we will show how to solve (13) $\hat{K}\tau=f$.

Assume that (13) has a solution τ, and suppose that ϕ is defined as a solution of

$$\Gamma\phi=\tau^* \tag{15}$$

where

$$\tau^*\equiv\tau-\psi(\tau),\quad \psi(\tau)\equiv\sum_{j=1}^{n}(\tau,\psi_j)\psi_j.$$

The existence of the solution ϕ of (15) is certified by Corollary 6. Then, on applying $\hat{K}$ on the both hand sides of (15) we have, because of (14),

$$\hat{K}\Gamma\phi=\hat{k}^*\phi=\hat{K}(\tau-\psi(\tau))$$

which turns out, because of (13), to be

$$\hat{k}^*\phi=f-\hat{K}\psi(\tau). \tag{16}$$

Since (16) is an equation of the first kind, it is solvable and its solution is given by

$$\phi=(I-K^*)^{-1}\Lambda^*f-(I-K^*)^{-1}\Lambda^*\hat{K}\psi(\tau)+\phi^{(0)} \tag{17}$$

where $\hat{k}^*\phi^{(0)}=0$ and Λ^* and K^* are operators defined as before corresponding to $\hat{k}^*$.

Hence, from (15) and (17), we have

$$\tau=\Gamma(I-K^*)^{-1}\Lambda^*f+\tau^{(0)} \tag{18}$$

where

$$\tau^{(0)}\equiv\{I-\Gamma(I-K^*)^{-1}\Lambda^*\hat{K}\}\psi(\tau)+\Gamma\phi^{(0)}.$$

With the help of Theorem 1, we can prove that

$$\hat{K}\tau^{(0)}=\hat{K}\Gamma\phi^{(0)}+\{I-\hat{K}\Gamma(I-K^*)^{-1}\Lambda^*\}\hat{K}\psi=0.$$

Hence, $\tau^{(0)}$ in (18) is a solution of

$$\hat{K}\tau^{(0)}=0. \tag{19}$$

Conversely, it is easy to show, with the help of Theorems 1 and 3 that τ given by (18) satisfies (13). Thus we have obtained

Theorem 7.

$$\hat{K}\tau=f \iff \tau=\Gamma(I-K^*)^{-1}\Lambda^*f+\tau^{(0)}. \tag{20}$$

Corollary 7. For ${}^{\forall}\phi\in H(L)$, we have the following identity:

$$\hat{K}\{\Gamma(I-K^*)^{-1}\Lambda^*\phi+\phi^{(0)}\}=\phi \tag{21}$$

where

$$\hat{K}\phi^{(0)}=0.$$

4. With regard to a general solution of a homogeneous equation $\hat{k}\tau^{(0)}=0$, we have the following theorem:

Theorem 8.

$$\hat{k}\tau^{(0)}=0 \iff \tau^{(0)}(t)=X(t)\sum_{n=-N}^{N+\nu''}p_nt^n \tag{22}$$

where p_n $(n=-N,-N+1,\cdots,N+\nu)$ are constants which satisfy the following equations:

$$\begin{aligned} &\sum_{n=0}^{m+N} \gamma_n p_{m-n} = \sum_{n=-N}^{N+\nu''} \alpha_{nm} p_n && (-N \leqq m \leqq -1) \\ &\sum_{n=r-\nu''}^{m-N-\nu''} \beta_n p_{m-n} = \sum_{n=-N}^{N+\nu''} \alpha_{nm} p_n && (0 \leqq m \leqq N) \\ &\sum_{n=r-\nu''}^{m-N-\nu''} \beta_m p_{m-n} = 0 && (N+1 \leqq m \leqq N+r). \end{aligned} \tag{23}$$

Remark 3. For the sake of brevity and definiteness, Theorem 8 is stated under the assumption that

$$k(t_0, t) \equiv \sum_{n=-N}^{N} k_n \frac{t_0^n}{t^{n+1}}, \quad r-\nu'' \leqq N.$$

For more general kernel $k(t_0, t)$, we have more complicated relations instead of (23).

In (23), β_n and γ_n are the coefficients of the Laurent expansion of $X(z)$ in the domain $r_2 < |z| < \infty$ and $0 < |z| < r_1$, respectively, where r_1 and r_2 are constants such that L exists in the domain $r_1 < |z| < r_2$. α_{nm} is given by $\pi i \alpha_{nm} \equiv k_m \int_L X(t) t^{n-m-1} dt$. Equation 23 is $2N+r+1$ simultaneous linear equation with regard to $2N+\nu''+1$ parameters p_n. Cases will be separated according to $r \gtreqless \nu''$.

References

[1] Muskehelishvili, N. I.: Singular Integral Equatio s. Noordhoff, Groningen, Holland (1953).

[2] Hayashi, Y.: Electromagnetic field in a domain bounded by a coaxial circular cylinder with slots. Rad. Lab. Memo, I, II, III (1963), and Proc. Japan Acad., **40**, 305 (1964).