68. *Electromagnetic Field in a Domain Bounded by Coaxial Circular Cylinders with Slots*

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1. In this paper, the exact solution of the Maxwell equations is derived by means of dual series equations and a singular integral equation with Cauchy kernel for a domain exterior to the inner member of a pair of coaxial circular cylinders where the outer member has a finite number of axial slots of infinite length and arbitrary width and in the presence of arbitrary axial line sources. This is a canonical form of some problems in radio engineering which became of interest recently. There are many works done on slotted cylinders [1], [2]. However, in these works, distribution of the field at the slot is usually assumed to be given or is approximated by a known distribution, say, by that of a static field at a slit on a plane. Recently, some work has been done [3] on waveguides by the method of singular integral equations.

In this paper, a compact formulation for the field components will be given which satisfies all required conditions, i.e., the boundary condition, the radiation condition, the edge condition at the edges of the slots and the continuity condition of the field at the slots.

2. Suppose that the expressions

$$r=a,\quad 0\leqq\phi\leqq 2\pi,\quad -\infty<z<\infty,$$
$$r=b,\quad \alpha_j<\phi<\beta_{j+1},\quad -\infty<z<\infty,$$
$$(0<a<b,\ \beta_j<\alpha_j<\beta_{j+1},\ \beta_{\nu+1}=\beta_1,\ j=1,2,\cdots,\nu)$$

represent a pair of coaxial circular cylinders of perfect conductivity, with ν slot specified by

$$r=b,\quad \beta_j<\phi<\alpha_j,\quad -\infty<z<\infty.$$

Without loss of generality, we can assume that there is one axial line source in the interior $(a\leqq r<b)$ at Q_i: $r=r_i$, $\phi=\phi_i$, and one axial line source in the exterior $(b<r)$ at $Q_e: r=r_e$, $\phi=\phi_e$, because fields for more sources are obtainable by the principle of superposition. In this case, it is easy to see that the Maxwell equations $\nabla\times E=-i\omega\mu H$, $\nabla\times H=i\omega\varepsilon E$ are equivalent to

$$\Delta u+k^2u=0 \tag{1}$$

with the boundary conditions

$$u=0\quad \text{at}\quad r=a\quad \text{and}\quad r=b,\ \alpha_j<\phi<\beta_{j+1}, \tag{2}$$

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and

$$\frac{\partial \mu}{\partial r}=0 \quad \text{at} \quad r=a \quad \text{and} \quad r=b,\ \alpha_j<\phi<\beta_{j+1} \tag{3}$$

where $k^2=\omega^2\varepsilon\mu$ and $\operatorname{Im} k \leqq 0$. u is the axial component $E_z(H_z)$ of $E(H)$ when the condition is (2) ((3)). It is also required that u satisfies the radiation condition at infinity and the edge condition at edges of slots.

As a solution of (1) which satisfies the radiation condition, u is represented as

$$u \equiv u_e = \sum_{n=-\infty}^{\infty} A_n H_n(kr)e^{in\phi}+f_e H_0(kR_e), \quad (b<r)$$
$$u \equiv u_i = \sum_{n=-\infty}^{\infty} \{B_n J_n(kr)+C_n H_n(kr)\}e^{in\phi}+f_i H_0(kR_i), \ (a<r<b) \tag{4}$$

where A's, B's and C's are unknown coefficients, and J_n and H_n are the Bessel function and the Hankel function of the second kind, respectively. R_e and R_i are the distances from Q_e and Q_i, respectively, and f_e and f_i are given constants which refer to the amplitudes of sources at Q_e and Q_i, respectively. At the slots, u and $\dfrac{\partial u}{\partial r}$ are required to be continuous, that is,

$$u_e=u_i \quad \text{and} \quad \frac{\partial u_e}{\partial r}=\frac{\partial u_i}{\partial r} \quad \text{at} \quad r=b,\ \beta_j<\phi<\alpha_j. \tag{5}$$

This requirement, together with (2), is equivalent to

$$u_i=0, \qquad (r=a,\ 0\leqq\phi\leqq 2\pi) \tag{6}$$
$$u_e=u_i, \qquad (r=b,\ 0\leqq\phi\leqq 2\pi) \tag{7}$$
$$u_e=0, \qquad (r=b,\ \alpha_j<\phi<\beta_{j+1}) \tag{8}$$
$$\frac{\partial u_e}{\partial r}=\frac{\partial u_i}{\partial r}, \quad (r=b,\ \beta_j<\phi<\alpha_j). \tag{9}$$

Similary, (5), together with (3), is equivalent to

$$\frac{\partial u_i}{\partial r}=0, \qquad (r=a,\ 0\leqq\phi\leqq 2\pi) \tag{10}$$
$$\frac{\partial u_e}{\partial r}=\frac{\partial u_i}{\partial r}, \quad (r=b,\ 0\leqq\phi\leqq 2\pi) \tag{11}$$
$$\frac{\partial u_e}{\partial r}=0, \qquad (r=b,\ \alpha_j<\phi<\beta_{j+1}) \tag{12}$$
$$u_e=u_i, \qquad (r=b,\ \beta_j<\phi<\alpha_j). \tag{13}$$

Conversely, if $u=E_z$ and $u=H_z$ are found, which satisfy the edge condition (which will be taken into account later) and (6)$\sim$(9) and (10)$\sim$(13), respectively, then the exact solutions E and H of the original equation are obtained.

3. First, we consider the "Dirichlet problem" (6)$\sim$(9). On substituting (4) into (6) and (7), and making use of the orthogonality $\{e^{in\phi}\}$ in $[0, 2\pi]$, one obtains simultaneous linear equations for B_n and C_n, from which B_n and C_n are determined in terms of A_n, $f_{en}\equiv f_e e^{-in\phi_e}$

and $f_{in}\equiv f_i e^{-in\phi_i}$. (For the sake of brevity, the formulas for B_n and C_n are not described here.) On the other hand, (8) and (9) turn out, by the substitution of (4) and the formulas for B_n and C_n obtained above, to be

$$\sum\{A_nH_n(kb)+f_{en}J_n(kb)H_n(kr_e)\}e^{in\phi}=0,\quad (\alpha_j<\phi<\beta_{j+1}),\tag{14}$$

$$\sum\frac{1}{D_n(a,b)}\{A_nH_n(ka)-f_{in}D_n(r_i,a)+f_{en}J_n(ka)H_n(kr_e)\}e^{in\phi}=0,\quad (\beta_j<\phi<\alpha_j)\tag{15}$$

where

$$D_n(r,\rho)\equiv J_n(kr)H_n(k\rho)-J_n(k\rho)H_n(kr)$$

respectively. (14) and (15) are the dual series equations for unknown coefficients A_n for the Dirichlet problems.

Corresponding to these results, those for the Neumann problem (10)$\sim$(13) are derived in the same way. Namely, (10) and (11) are reduced to simultaneous linear equations, from which the formulas for B_n and C_n are obtained in terms of A_n. Also, (12) and (13) turn out to be

$$\sum\{A_nH_n'(kb)+f_{en}J_n'(kb)H_n(kr_e)\}e^{in\phi}=0,\quad (\alpha_j<\phi<\beta_{j+1}),\tag{14$'$}$$

$$\sum\frac{1}{T_n(a,b)}\{A_nH_n'(ka)-f_{in}L_n(r_i,a)+f_{en}J_n'(ka)H_n(kr_e)\}e^{in\phi}=0,\quad (\beta_j<\phi<\alpha_j)\tag{15$'$}$$

respectively, where the primes indicate derivatives of J_n and H_n with respect to their arguments and

$$T_n(r,\rho)\equiv J_n'(kr)H_n'(k\rho)-J_n'(k\rho)H_n'(kr),$$
$$L_n(r,\rho)\equiv J_n(kr)H_n'(k\rho)-J_n'(k\rho)H_n(kr).$$

Conversely, if A's are determined so that they satisfy (14) and (15) ((14)$'$ and (15)$'$), and if B's and C's are determined by the formulas obtained before, then u's defined by (4) are easily shown to be the exact solution for (6)$\sim$(9) ((10)$\sim$(13)).

4. The left-hand side of (15) can be considered to be a Fourier expansion of an unknown function $2\pi\tau$ in $[0,2\pi]$ which is identically zero in (β_j,α_j). Hence we have

$$A_n=\frac{D_n(a,b)}{H_n(ka)}\int_{\text{wall}}\tau(\theta)e^{-in\theta}d\theta+\frac{1}{H_n(ka)}\{f_{in}D_n(r_i,a)-f_{en}J_n(ka)H_n(kr_e)\}.\tag{16}$$

On substituting (16) into (14), and assuming that the summation is interchangeable with the integration, one has

$$\int_L\tau(\theta)\sum_{n=-\infty}^{\infty}S_ne^{in\Theta}d\theta=g(\phi),\quad \Theta\equiv\phi-\theta\tag{17}$$

where L is the wall; that is, the union of intervals (α_j,β_{j+1}), and

$$S_n\equiv\frac{H_n(kb)}{H_n(ka)}D_n(a,b)$$

$$g(\phi)\equiv\sum\frac{e^{in\phi}}{H_n(ka)}\{f_{en}H_n(kr_e)D_n(ab)-f_{in}H_n(kb)D_n(r_i,a)\}.\tag{18}$$

In a way similar to this, A_n for the Neumann problem is represented from (14)′, as

$$A_n=\frac{1}{H_n'(kb)}\left\{\int_{\text{slot}}\tau(\theta)e^{-in\theta}d\theta-f_{en}J_n'(kb)H_n(kr_e)\right\} \tag{16$'$}$$

where slot means a union of intervals (β_j, α_j), and $2\pi\tau(\phi)$ is an unknown function defined by the left hand side of (14)′. From (15)′ and (16)′, one derives

$$\int_L \tau(\theta)\sum S_n e^{in\theta}d\theta=g(\phi) \tag{17$'$}$$

where L is the slot and

$$S\equiv\frac{H_n'(ka)}{H_n'(kb)T_n(a, b)}$$

$$g(\phi)\equiv\sum e^{in\phi}\left\{f_{in}\frac{L_n(r_i, a)}{T_n(a, b)}-f_{en}\frac{H_n(kr_e)}{H_n'(kb)}\right\}. \tag{18}$$

Thus we have reduced the dual series equations (14) and (15), ((14)′ and (15)′) to an integral equation (17) ((17)′). Note that (17) and (17)′ are the same, though L, S_n and g should be correctly understood for each case. Conversely, if τ is a solution of (17) ((17)′), then A's defined by (16) ((16)′) are easily proved to satisfy (14) and (15) ((14)′ and (15)′). Thanks to the same form of (17) and (17)′, we can solve these integral equations for both cases of $u=E_z$ and $u=H_z$ in the same way.

5. By the well-known relations: $J_{-n}=(-1)^n J_n$ and $H_{-n}=(-1)^n H_n$, one can show that $S_n=S_{-n}$. Furthermore, one has

$$S_n=\frac{-ic}{2n}\{1+s_{|n|}\},\quad c=\begin{cases}\dfrac{2}{\pi}, & \text{for (18)}\\ 2\pi(kb)^2 & \text{for (18)}'\end{cases} \tag{19}$$

where $s_{|n|}$ are known, and are small quantities if $0<N<n$, where N is a constant which is determined by ka and kb.

On substituting (19) into (17) ((17)′), one has

$$\int_L \tau(\theta)\left\{S_0+\frac{ic}{2}\log(2-2\cos\Theta)-ic\sum_{n=1}^{N}\frac{s_n}{n}\cos n\Theta\right\}d\theta=g(\phi). \tag{20}$$

Since (20) is not solvable, we differentiate it with respect to ϕ, obtaining

$$\int_L \tau(\theta)\left\{\frac{i}{2}-\frac{\sin\Theta}{2-2\cos\Theta}\right\}d\theta=\int_L \tau(\theta)k(\phi, \theta)d\theta+f(\phi), \tag{21}$$

where

$$k(\phi, \theta)\equiv\frac{i}{2}+\sum_{n=1}^{N}\left\{-1+\frac{2in}{c}S_n\right\}\sin n\Theta$$

$$f(\phi)\equiv\frac{i}{c}\frac{dg(\phi)}{d\phi}.$$

Note that the differentiation and integration of (20) are interchangeable because the integrand of (20) has only the logarithmic singularlity.

Suppose that $\pm = be^{i\theta}$ and $t_0 = be^{i\phi}$ are points on L; then (21) is shown to be equivalent to a singular integral equation with Cauchy kernel,

$$\int_L \tau(t)\left\{\frac{1}{t-t_0} - k(t_0, t)\right\} dt = f(t_0), \tag{23}$$

where $\tau(t), f(t)$ and $itk(t_0, t)$ are functions corresponding to $\tau(\theta), f(\theta)$ and $k(\phi, \theta)$ respectively.

Remark 1. $\tau(\phi)$ is defined by (15') ((14)') which is originally equal to (9) ((12)). Hence, τ is proportional to $(H_\phi)_e - (H_\phi)_i$, $((E_\phi)_e)$. These quantities are known to have singularities of $o(\rho^{-1/2})$ at the edges of the slots, where ρ is the distance from the edges [4]. This is the required edge condition. Hence, we are looking for a solution τ of (23) which has a singularity of $o(\rho^{-1/2})$ at the end points of L.

Remark 2. Since (23), or (21), has been derived from (20) by the differentiation with regard to ϕ, a solution of (23) does not necessarily satisfy (20)). However, since the general solution of (23) is composed of a particular solution of (23) and a general solution of a homogeneous equation (i.e., eq. (23) with $f(t_0) \equiv 0$), we are able to pick the solution of (20) up among those of (23), with a pertinent choice of values for arbitrary constants in a general solution of the homogeneous equation.

The integral equation (23) is solved by the application of the theory obtained by the author [5], as follows:

Def.

(i) End points of L, i.e., those points $be^{i\alpha_j}$ and $be^{i\beta_j}$ $(j=1, 2, \cdots, \nu)$ are denoted as c_l $(l=1, 2, \cdots, 2\nu)$ in any order.

(ii) $X(z) \equiv 1\sqrt{\prod_{l=1}^{2\nu}(z-c_l)}$

(iii) $K_1(t_0, t) \equiv \dfrac{1}{\pi^2}\displaystyle\int_L \frac{X(\zeta)}{X(t)}\left\{\frac{1}{\zeta-t_0} - k(t_0, \zeta)\right\}\left\{\frac{1}{t-\zeta} - \lambda(\zeta, t)\right\} d\zeta$

where $\lambda(t_0, t)$ is a function such that $\|K_1\| < 1$.

(iv) $f_0(t_0) \equiv f(t_0)$, $f_n(t_0) \equiv \displaystyle\int_L K_1(t_0, t) f_{n-1}(t) dt$.

(v) $\Lambda\varphi(t_0) \equiv \dfrac{X(t_0)}{\pi i}\displaystyle\int_L^* \left\{\frac{1}{t-t_0} - \lambda(t_0, t)\right\}\frac{\varphi(t)}{X(t)} dt$.

In this terminology, the general solution of (23) is given by

$$\tau(t) = \Lambda\sum_{n=0}^{\infty} f_n(t_0) + \tau^{(0)},$$

$$\tau^{(0)}=X(t)\sum_{n=-N}^{N+\nu}p_n t^n : \hat{k}\tau^{(0)}=0, \tag{24}$$

where $p_n(n=-N, -N+1,\cdots, N+\nu)$ are constants which are determined by the relations

$$\sum_{n=0}^{m+N} \gamma_n p_{m-n}=\sum_{n=-N}^{N+\nu}\alpha_{nm}p_n \quad (-N\leqq m\leqq -1)$$

$$\sum_{n=-\nu}^{m-N-\nu} \beta_n p_{m-n}=\sum_{n=-N}^{N+\nu}\alpha_{nm}p_n \quad (0\leqq m\leqq N)$$

β_m and α_m are coefficients of the Laurent expansion of $X(z)$ at $z=\infty$ and $z=0$, respectively and $\alpha_{nm}\equiv\dfrac{k_m}{\pi i}\displaystyle\int_L X(t)t^{n-m-1}dt$, where

$$k_m\equiv\begin{cases} \dfrac{1}{2}-\dfrac{im}{c}S_m; & m>0 \\ \dfrac{1}{2} & ;\ m=0 \\ -\dfrac{1}{2}-\dfrac{im}{c}S_m; & m<0. \end{cases}$$

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