89. *On Endomorphism with Fixed Element on Algebra*

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In this note, we shall consider endomorphisms with a fixed element on general algebra. For simplicity, we consider an endomorphism T on a semigroup S. Let us suppose $T(a)=a$. We denote the kernel of the endomorphism T^n, i.e. the set of all elements x such that $T^n(x)=a$ by $\ker(T^n)$, and the image $T^n(S)$ by $\mathrm{Im}(T^n)$. If for some n, $\ker(T^n)=\ker(T^{n+1})$, then T is called a *γ-endomorphism.* Then we have $\ker(T^n)=\ker(T^{n+1})=\cdots=\ker(T^m)=\cdots$, where $n\leq m$. The least number n satisfying $\ker(T^n)=\ker(T^{n+1})$ is called the *order* of T.

Let n be the order of T, then for $n\leq m$, we have

$$\ker(T^m)\cap\mathrm{Im}(T^m)=(a). \tag{1}$$

To prove it, let $x\in\ker(T^m)\cap\mathrm{Im}(T^m)$, then we have $T^m(x)=a$ and $x=T^m(y)$ for some $y\in S$. Hence $T^{2m}(y)=T^m(x)=a$, so $y\in\ker(T^{2m})=\ker(T^m)$. Therefore $T^m(y)=a$, and we have $x=a$.

Conversely, the least number m satisfying (1) is the order of T.

It is sufficient to prove that (1) implies $\ker(T^m)=\ker(T^{m+1})$. In general, we have

$$(a)\subset\ker(T)\subset\ker(T^2)\subset\cdots. \tag{2}$$

To prove the inclusion $\ker(T^{m+1})\subset\ker(T^m)$, let $x\in\ker(T^{m+1})$. Then $T^{m+1}(x)=a$ and so $T(T^m(x))=a$.

Hence $T^m(x)\in\ker(T)$. On the other hand, (1) and (2) imply $\ker(T)\cap\mathrm{Im}(T^m)=(a)$. Therefore $T^m(x)\in\ker(T)\cap\mathrm{Im}(T^m)=(a)$, and we have $T^m(x)=a$. This means $x\in\ker(T^m)$.

Therefore we have the following

THEOREM. *Let T be a γ-endomorphism of order n on a semigroup S, and $T(a)=a$. Then for $m\geq n$,*

$$\ker(T^m)\cap\mathrm{Im}(T^m)=(a). \tag{1}$$

Conversely, the least number m satisfying (1) *is the order of T.*

A similar result for linear spaces has been stated by several authors, for example, by M. Audin [1], and for the case of groups by H. Ramalho [2].

References

[1] M. Audin: Sur les équations linéaires dans un espace vectoriel. Alger, Mathématique, **4**, 5-71 (1957).

[2] M. Ramalho: Sur quelques théorèmes de la théorie des groupes. Rev. Fac. Ciências Lisboa 2, série A, **8**, 333-337 (1961).