88. *On Subadditive Functionals and Linear Functionals on Abelian Group*

By Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1964)

Recently D. Milman [2] has proved an interesting theorem relating with Hahn-Banach theorem. In this note, we shall consider his result on an Abelian group.

Let G be an Abelian group, and consider a subadditive functional $p(x)$ on G, i.e. $p(x+y)\leq p(x)+p(y)$ and $p(0)=0$.

Define an order $p_2\prec p_1$ if $p_2(x)\leq p_1(x)$ for every $x\in G$. We have a well known theorem by Aumann: There is a linear functional $f(x)$, i.e. $f(x+y)=f(x)+f(y)$ such that $f\prec p$ for each subadditive p (see K. Iséki [1]). Now we have the like of Milman result.

Theorem. *Let $p(x)$ be a subadditive functional, not linear functional. Then there is at least one minimal element for p on the order $\prec$, and its element is linear. The set consisting of all elements of linear functionals f such that $f\prec p$ coincides with the total set of minimal elements for p.*

To prove Theorem, we shall use a similar technique by D. Milman.

Proof. Since $p(x)$ is not linear, there are two elements x_1, y_1 such that $p(x_1+y_1)<p(x_1)+p(y_1)$. Let H be the subgroup generated by x_1, y_1, then by Aumann theorem, there is a linear functional $f(x)$ on H such that $f(x)\leq p(x)$ for $x\in H$. Put

$$p_1(x)=\inf_{y\in H}\{f(y)+p(x-y)\}$$

for $x\in G$. Then $-p(-x)\leq f(y)+p(x-y)$ implies $-p(-x)\leq p_1(x)\leq p(x)$. Therefore $p_1(x)$ is well-defined on G. Further, we have $p_1(x+y)\leq p_1(x)+p_1(y)$ and $-p(-y)\leq f(y)\leq p(y)$ for $y\in H$ implies $p_1(0)=0$. On the other hand, from

$$f(x_1)+f(y_1)=f(x_1+y_1)\leq p(x_1+y_1)<p(x_1)+p(y_1)$$

and $f(x_1)\leq p(x_1)$, $f(y_1)\leq p(y_1)$, for example, we have $p(x_1)-f(x_1)>0$. Hence

$$p_1(x_1)\leq f(x_1)+p(x_1-x_1)=f(x_1)<p(x_1),$$

and so $p_1\neq p$ and $p_1\prec p$.

If $\{p_\alpha\}$ is totally ordered set, then $p=\inf_\alpha p_\alpha$ is well-defined and subadditive on G. Hence at least one minimal element p exists by Zorn's lemma. Suppose that p is not linear, then there is a subadditive functional p_1 such that $p_1\prec p$ by the fiirst step of the proof. This is a contradiction.

If f is linear and $p_1\prec f$, then we have $f(x)=-f(-x)\leq -p_1(-x)\leq$

$p_1(x) \leq f(x)$ and so $p_1 = f$, which completes the proof.

References

[1] K. Iséki: On existence of linear functionals on Abelian groups. Proc. Japan Acad., **40**, 70 (1964).
[2] D. Milman: Decomposition of non-linear functional and its linear extension. Izvestia Akad. Nauk. U.S.S.R., **27**, 1189–1210 (1963).