# 86. A Note on Statistical Metric Spaces

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**1. Introduction.** Statistical metric space is a space in which a probability distribution function $F_{pq}(x)$ is associated with each pair $(p, q)$ of its points, while in a metric space a definite non-negative number is made to correspond to each pair. Some restrictions like the axioms of distances in metric spaces should be placed on the distribution functions of statistical metric spaces. One can find a typical formulation of these conditions together with a brief history of this kind of spaces and the references in [1] by B. Schweizer and A. Sklar. Several interesting results obtained by these authors and by E. Thorp are found also in [2], [3], [4], and [5]. In the series of these treatises, an axiom

$$F_{pr}(x+y) \geqq T(F_{pq}(x), F_{qr}(y)) \qquad (1)$$

plays an important role throughout, where $T(u, v)$ is a function defined in the unit square and satisfies conditions such as

$$T(a, b) = T(b, a),\ T(a, 1) = a,\ T(0, 0) = 0 \quad \text{etc.}$$

This axiom corresponds to the triangular inequality in the metric spaces. Now here we have an important problem how to define a topology in a statistical metric space $S$. In the papers listed above, a topological structure of $S$ is given by the system of neighbourhoods $\{N_p(\varepsilon, \lambda)\}$, where

$$N_p(\varepsilon, \lambda) = \{q;\ F_{pq}(\varepsilon) > 1-\lambda\}.$$

This scheme is closely connected to the convention

$$F_{pp}(x) \equiv H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqq 0 \end{cases}.$$

It is to be noted that $S$ is a Hausdorff space with the system of neighbourhoods $\{N_p(\varepsilon, \lambda)\}$ and $F_{pq}(x)$ is continuous with respect to $(p, q)$. But some additional assumptions other than the conditions on $t$-functions are necessary to obtain the latter result.

In this note, we first define a topology in the space of distribution functions and then reflecting this structure, a topology will be introduced into $S$ to the effect that $S$ is a Hausdorff space and $F_{pq}$ is continuous with respect to $(p, q)$ without $t$-functions.

**2. Topology in the space of distribution functions.** $\mathfrak{F}$ is the set of all functions $F(x)$ of a real variable $x$ having properties i) non-negative valued, ii) monotone increasing, iii) right continuous, iv) $F(x)=0$ for $x<0$, and v) $F(x) \to 1$ as $x \to +\infty$.

In what follows, $F(x), G(x), \cdots$ will be abbreviated as $F, G, \cdots$ and $F \neq G$ means $F(x) \not\equiv G(x)$.

DEFINITION 1. Suppose $\eta$ is given and $0<\eta<1$.

i) If a value of $x$ such that $F(x)=\eta$ is uniquely determined, this value is $x(F, \eta)$.

ii) If $F(x)=\eta$ for $x$ belonging to an interval $(x_1, x_2)$ and $F(x)\neq\eta$ outside, $x(F, \eta)=\frac{1}{2}(x_1+x_2)$.

iii) If $F(x)\neq\eta$ for each $x$, greatest lower bound of $x$ such that $F(x)>\eta$ should be taken as $x(F, \eta)$.

The condition $0<\eta<1$ will not be written explicitly in the following.

DEFINITION 2. Being given positive numbers $\varepsilon$ and $\eta$, we put $V(\varepsilon, F)=\{G(x); \exists t_G(0<t_G<1), \forall\eta, |x(F, \eta)-x(G, \eta)|<t_G\varepsilon\}$, where $t_G$ is associated with each $G(x)$ for each $V(\varepsilon, F)$. $V(\varepsilon, F)$ is an $\varepsilon$-neighbourhood of $F$ in $\mathfrak{F}$.

We obtain the following propositions immediately:

PROPOSITION 1. *If* $0<\varepsilon_1<\varepsilon_2$ *then* $V(\varepsilon_1, F)\subset V(\varepsilon_2, F)$.

PROPOSITION 2. *If* $F_1 \in V(\varepsilon_1, F)$ *and* $F_2 \in V(\varepsilon_2, F_1)$ *then*

$$F_2 \in V(\varepsilon_1+\varepsilon_2, F).$$

Now we can prove an important proposition.

PROPOSITION 3. $\mathfrak{F}$ *with the system of neighbourhoods* $\{V(\varepsilon, F)\}$ *is a Hausdorff space.*

PROOF. i) It is evident that $F \in V(\varepsilon, F)$.

ii) If $F_1 \in V(\varepsilon, F)$ then by definition there exists $t_{F_1}$ such that

$$0<t_{F_1}<1, \forall\eta, |x(F_1, \eta)-x(F, \eta)|<t_{F_1}\varepsilon.$$

Put $\varepsilon_1=(1-t_{F_1})\varepsilon$, then $\varepsilon_1>0$ and for $G \in V(\varepsilon_1, F_1)$ there exists $t_G$ such that

$$0<t_G<1, \forall\eta, |x(F, \eta)-x(G, \eta)|<t_G\varepsilon_1.$$

Now putting $t_{F_1}+t_G-t_{F_1}t_G=t'_G$ we have $0<t'_G<1$ so that

$$\forall\eta, |x(F, \eta)-x(G, \eta)|<t'_G\varepsilon,$$

this proves that $V(\varepsilon_1, F_1)\subset V(\varepsilon, F)$.

iii) If $\varepsilon_1\neq\varepsilon_2$, suppose $\varepsilon_1<\varepsilon_2$. It is an immediate consequence of the Proposition 1 that

$$V(\varepsilon_1, F)\cap V(\varepsilon_2, F)=V(\varepsilon_1, F).$$

iv) In case where $F\neq G$, there exists $\eta_1$ such that

$$x(F, \eta_1)\neq x(G, \eta_1).$$

For two positive numbers $\varepsilon_1$ and $\varepsilon_2$ satisfying

$$\varepsilon_1+\varepsilon_2<|x(F, \eta_1)-x(G, \eta_1)|,$$

it holds that

$$V(\varepsilon_1, F)\cap V(\varepsilon_2, G)=\phi.$$

**3. Topology of $S$.** The mapping from $S\times S$ into $\mathfrak{F}$ should be subjected to the following axioms:

**Axiom I.** An element $F_{pq}(x)$ of $\mathfrak{F}$ is associated with each element $(p, q)$ of $S\times S$.

**Axiom II.** $F_{pq}(x) \equiv F_{qp}(x)$.

**Axiom III.** If $p \neq q$ then there exists a point $r$ of $S$ such that $F_{pr} \neq F_{qr}$.

DEFINITION 3. Being given a positive number $\varepsilon$ we put

$$W(\varepsilon, p) = \{q; \forall r, F_{qr} \in V(\varepsilon, F_{pr})\},$$

and $W$ is the family of $W(\varepsilon, p)$ for all $\varepsilon(>0)$ and $p(\in S)$.

DEFINITION 4. For each point $p$ of $S$, the neighbourhood system $\{U(p)\}$ of $p$ is the family consisting of all finite intersections of $W$'s belonging to $W$ and containing $p$.

PROPOSITION 4. *$S$ with the system of neighbourhoods $\{U(p)\}(p \in S)$ is a Hausdorff space.*

PROOF. Three conditions i) $p \in U(p)$, ii) if $q \in U(p)$ there exists $U(q)$ contained in $U(p)$, and iii) there exists $U(p)$ contained in $U_1(p) \cap U_2(p)$ are easily verified.

iv) If $p \neq q$ it is certain by Axiom III that there exists a point $r$ for which $F_{pr} \neq F_{qr}$. Proposition 3 enables us to choose a positive number $\varepsilon$ for this $r$ such that

$$V(\varepsilon, F_{pr}) \cap V(\varepsilon, F_{qr}) = \phi \tag{2}$$

Suppose now that $W(\varepsilon, p)$ and $W(\varepsilon, q)$ have a point $s$ in common, then for these $s$ and $r$ we have

$$F_{sr} \in V(\varepsilon, F_{pr}) \cap V(\varepsilon, F_{qr})$$

which contradicts (2). Thus, taking $W(\varepsilon, p)$ and $W(\varepsilon, q)$ respectively as $U(p)$ and $U(q)$, we conclude

$$U(p) \cap U(q) = \phi.$$

**4. Continuity of $F_{pq}$ with respect to $(p, q)$.**

PROPOSITION 5. *$F_{pq}$ is continuous with respect to $(p, q)$.*

PROOF. Being given a positive number $\varepsilon$ and $(p, q)(\in S \times S)$, let $\varepsilon_1 = \frac{1}{2}\varepsilon$ and take neighbourhoods of $p$ and $q$ as

$$U(p) = W(\varepsilon_1, p), \quad U(q) = W(\varepsilon_2, q).$$

Then for $p' \in U(p)$ and $q' \in U(q)$ we have

$$\forall r, F_{p'r} \in V(\varepsilon_1, F_{pr}) \tag{3}$$

$$\forall s, F_{q's} \in V(\varepsilon_1, F_{qs}). \tag{4}$$

Let $r = q$ in (3) and $s = p'$ in (4). Then by Proposition 2 we obtain

$$F_{p'q} \in V(\varepsilon_1, F_{pq}), \quad F_{p'q'} \in V(\varepsilon_1, F_{p'q})$$

and consequently

$$F_{p'q'} \in V(\varepsilon, F_{pq})$$

which proves that $F_{pq}$ is continuous with respect to $(p, q)$.

**Remarks.** We intentionally disregarded the convention which specifies $F_{pp}(x)$ as $H(x)$ (unit step function). Though there arises no difficulty from this convention, some modifications would prove preferable if the definition of $W(\varepsilon, p)$ appeared too restrictive in the case where $F_{pp}(x) \equiv H(x)$. In this case, introducing a new parameter $\lambda$, $V(\varepsilon, F)$ may be substituted by

$$V(\varepsilon, \lambda, F)=\{G(x); \exists t_G, \forall \eta, 0<\eta \leqq \lambda<1, |x(F, \eta)-x(G, \eta)|<t_G \varepsilon\},$$

so $W(\varepsilon, p)$ too by $W(\varepsilon, \lambda, p)$ and so on. It is not difficult to see that main results obtained above remain unchanged through such modifications.

## References

[1] B. Schweizer and A. Sklar: Statistical metric spaces. Pacific J. of Math., **10**, no. 1 (1960).

[2] B. Schweizer, A. Sklar, and E. Thorp: The metrization of statistical metric spaces. Pacific J. of Math., **10**, no. 2 (1960).

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[5] B. Schweizer and A. Sklar: Triangle inequalities in a class of statistical metric spaces. J. of London Math. Soc., no. 152 (1963).