84. C^* Algebra and its Extension as the Set of Observables

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§1. **Introduction.** The operator A having the bounded usual expectation value $\langle \Phi, A\Psi \rangle$ for any two states Φ and Ψ is a bounded operator.

These bounded operators construct C^* algebra which is considered as the set of observables. It seems to us that the name of observables is derived from the reason such that it has always bounded usual expectation value. (Namely, it can be observed.)

On the other hand the most of the quantities appearing in quantum field theory are unbounded operators such as field function, creation operator and annihilation operator etc. The set of unbounded operators is also investigated by John Von Neumann [2], [3]. But the topology to give the relation between the bounded operators and the unbounded operators is not to be seen in his work. Using spectral decomposed form we can obtain the series of bounded operators approached to the self adjoint unbounded operator. But it is difficult to treat concretely the unbounded operators using above method. To treat unbounded operators concretely, weak closure of the set of bounded operators is used by R. Kastler and K. Haag in [1]. But his topology is too strong to extend the set of observables. Using the weak topology related to a fixed dense subspace, this difficulty is eliminated temporarily and unnaturally [3].

In this paper, we show the defect of extension in [1], and extend truely the set of observables by using E. R. Integral [4], considering the various methods.

§2. **Observable.** Abstruct C^* algebra is the essential tool of axiomatic relativistic quantum field theory [1]. At the first step, let's show the construction of it.

Let B_i denote the sets contained in 4 dimensional Minkovski space. Let $\mathfrak{A}(B)$ denote C^* algebra related to B.

$\mathfrak{A}(B)$ has the following properties:

(1) To every relative compact open set B, one $\mathfrak{A}(B)$ is corresponded.

(2) If B_1 contains B_2, then $\mathfrak{A}(B_1)$ contains $\mathfrak{A}(B_2)$.

(3) If B_1 and B_2 are completely space like, then $\mathfrak{A}(B_1)$ and $\mathfrak{A}(B_2)$ are mutually commutative.

Let $\mathfrak{A}$ denote the completion of $\bigcup_B \mathfrak{A}(B)$ in quasi norm. $\mathfrak{A}$ is evidently C^* algebra.

(4) $\mathfrak{A}$ is the Lorentz covariant space.

(5) $\mathfrak{A}$ is primitive.

Let $\mathfrak{H}$ denote the sub-Hilbert space of Von Neumann's direct product space whose bases are consisted of the states $\prod_k \otimes \varphi(n_k) = \prod_k [a(\boldsymbol{k})]^{n(\boldsymbol{k})} \cdot \prod_k \otimes \varphi(0_k)$, where $n(\boldsymbol{k})$ is the non-negative integer valued function of $\boldsymbol{k}$. Namely, the continuous representation is not contained in this space.

Let $R(\mathfrak{A})$ denote the representation of $\mathfrak{A}$ by the space of bounded operators defined on $\mathfrak{H}$.

Using this representation the expectation value $\psi(A)=(\psi, A\psi)$ is defined.

For a fixed A contained in $R(\mathfrak{A})$ and for any pair (ψ_1, ψ_2) of the elements in $\mathfrak{H}$, $(\psi_1, A\psi_2)$ is bounded and defined. But, for A not contained in $R(\mathfrak{A})$ there is a pair (ψ_1, ψ_2) of the elements in $\mathfrak{H}$ such that $(\psi_1, A\psi_2)$ constructed from this pair is not bounded or is not defined. Using the self adjoint operator $a^+(\mathbf{0})+a(\mathbf{0})$ which is one term of the well-known representation of the field function $\varphi(\boldsymbol{x})$, let's show this aspect by two examples. Our purpose of this paper is to avoid this difficulty.

Example 1. Suppose that $\prod_k \otimes \psi(n_k)$ is direct product of the following element $\psi(n_k)$ in $\mathfrak{H}_k$:

(1) $\psi(n_k)=\varphi(0_k)$ for $\boldsymbol{k} \neq \mathbf{0}$,

(2) $\psi(n_0)=\sum_{n=0}^{\infty}(1/(n+1)^{3/4})\varphi(n_0)$.

Here, $\varphi(n_0)$ is the element in $\mathfrak{H}_0$ with norm 1 having the orthogonality for different n.

This state can be decomposed in

$$\sum_{n=0}^{\infty}(1/(n+1)^{3/4})(\prod_{k\neq 0} \otimes \varphi(0_k)) \otimes \varphi(n_0).$$

Let $a^+(\mathbf{0})$ and $a(\mathbf{0})$ denote the operators satisfying the relations

$$a^+(\mathbf{0})(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(n_0)=\sqrt{n+1}\cdot(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(n+1_0),$$
$$a(\mathbf{0})(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(n_0)=\sqrt{n}\cdot(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(n-1_0).$$

Then

$$\begin{aligned}(a^+(\mathbf{0})+a(\mathbf{0}))\cdot\textstyle\sum_{n=0}^{\infty}(1/(n+1)^{3/4})(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(n_0)\\ =\textstyle\sum_{n=1}^{\infty}(1/n^{1/4}+(n+1)^{1/2}/(n+2)^{3/4})(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(n_0)\\ +(1/2^{3/4})(\prod_{k\neq 0}\otimes\varphi(0_k))\otimes\varphi(0_0).\end{aligned}$$

$$\begin{aligned}(\textstyle\prod_k\otimes\psi(n_k),\ (a^+(\mathbf{0})+a(\mathbf{0}))\prod_k\otimes\psi(n_k))\\ =1/2^{3/4}+\textstyle\sum_{n=1}^{\infty}(1/n^{1/4}+(n+1)^{1/2}/(n+2)^{3/4})/(n+1)^{3/4}=\infty.\end{aligned}$$

Example 2. Suppose that $\prod_k \otimes \tilde{\psi}(n_k)$ is direct product of the following element $\psi(n_k)$ in $\mathfrak{H}_k$:

(1) $\tilde{\psi}(n_k)=\varphi(0_k)$ for $\boldsymbol{k}\neq\mathbf{0}$,

(2) $\tilde{\psi}(n_0)=\sum_{n=0}^{\infty}(-1)^n(1/(n+1)^{3/4})\varphi(n_0)$.

Then

$$(\textstyle\prod_k\otimes\psi(n_k),\ (a^+(\mathbf{0})+a(\mathbf{0}))\prod_k\otimes\tilde{\psi}(n_k))$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}(1/n^{1/4}+(n+1)^{1/2}/(n+2)^{3/4})/(n+1)^{3/4}-1/2^{3/4},$$

where $\prod_k \otimes \psi(n_k)$ is the state defined in Example 1.

In spite of the above examples, the use of C^* algebra $\mathfrak{A}$ as the set of observable is not necessary suitable, because the usual field functions are not bounded operators.

Hence the extension of the concept of observables is needed.

For this purpose Von Neumann construct the set of operators X such that $X\eta R(\mathfrak{A})$. Namely, X is the set of all operators which are commuted with any unitary operators contained in the commutant $R(\mathfrak{A})'$. The purpose of this paper is to find the constructing method of X from $R(\mathfrak{A})$ [3], [7].

For the extension of the set of observables, there are two sorts of different methods. Let's show them. Suppose that the expectation value $(\psi_1, A\psi_2)$ is defined for a pair (ψ_1, ψ_2) contained in $D_1\times D_2(\subset\mathfrak{H}\times\mathfrak{H})$, namely $D_1\times D_2$ is the domain of $(\psi_1, A\psi_2)$.

(1) Take suitable dense subsets D_1 and D_2 in $\mathfrak{H}$. And let's define the observables as follows:

The set of observables is the set of A satisfied the condition such that $(\psi_1, A\psi_2)$ is finite and definite for any pair $(\psi_1, \psi_2)\in D_1\times D_2$. Here, the unique determination of $D_1\times D_2$ is an important problem. (Hence Von Neumann favors the factor of type II_1.) [3]

For example, $D_1=D_2=\{\sum_{i=1}^{m}a_i\prod_k\otimes\varphi(n_k^{(i)});\ n_k^{(i)}$ is bounded for any fixed $k\}$. Then, the creation operator, the annihilation operator and $\varphi(x)*\rho(x)$ are considered as the observables. Here $\varphi(x)$ is a field function and $\rho(x)$ is the element in C_0^∞. Prof. K. Kunugi has showed the axiom based on the case (1) at Kanseigakuin University. But the case (2) is more suitable than the case (1), because it can obtain a value $(\psi_1, A\psi_2)$ for any pair (ψ_1, ψ_2) (not necessarily finite).

(2) Let's take $D_1=D_2=\mathfrak{H}$.

The set of observables is the set of A such that for any $\psi\in\mathfrak{H}$ and $\psi_2\in\mathfrak{H}$ $(\psi_1, A\psi_2)$ is defined by using some rule of the choice of conditional convergent series. To determine its rule, E. R. Integral investigated by Prof. K. Kunugi and others is most suitable, because we can connect to the observation theory by it.

If we use distribution's theory instead of E. R. Integral, the singularity can be fully avoided, but the connection to the observation theory is not satisfied. In §4 we show this.

§3. Weak extension. In [1] extension of $R(\mathfrak{A})$ by using "weak topology" is obtained, because "weak topology" can be related to the observation theory. But if we regard the weak topology used by R. Kastler and K. Haag as the ordinary one, it seems to us that by using it we cannot extend observables to useful direction. Hereafter, we show this.

At the first step let's define the ordinary weak topology. Suppose that A is contained in $R(\mathfrak{A})$, φ_i and ψ_i $(i=1, 2, \cdots, n)$ are contained in $\mathfrak{H}$, ε is an arbitrary fixed positive number and n is an arbitrary fixed positive integer.

Definition 1. 1) The weak topology τ_1 (in [1]) is one defined by using the family of all possible neighbourhoods $\{U(A; \psi_1, \cdots, \phi_n, \varepsilon)\}$ such that $U(A; \psi_1, \cdots, \psi_n, \varepsilon)=\{X; X\in R(\mathfrak{A}),\ |\psi_i(A)-\psi_i(X)|<\varepsilon, i=1, \cdots, n\}$.

2) The weak topology τ_2 is one defined by using the family of all possible neighbourhoods $\{U(A; \psi_1, \cdots, \psi_n, \varphi_1, \cdots, \varphi_n, \varepsilon)\}$ such that

$$U(A; \psi_1, \cdots, \psi_n, \varphi_1, \cdots, \varphi_n, \varepsilon) =\{X; X\in R(\mathfrak{A}),\ |\langle\psi_i\, A\varphi_i\rangle-\langle\psi_i, X\varphi_i\rangle|<\varepsilon,\ i=1, \cdots, n\}.$$

Let α and β denote the non negative numbers such that

$$\alpha=\sup\{|(x, Ay)|;\ ||x||=||y||=1\}$$
$$\beta=\sup\{|(x, Ax)|;\ ||x||=1\}.$$

Here x, y are contained in $\mathfrak{H}$ and A is a bounded operator. We obtain the following

Theorem 1. $\beta\leqq\alpha\leqq 2\beta$ [8].

Proof. $\beta\leqq\alpha$ is evident.

Since

$$(x, Ay)=(1/4)\{(x+y, A(x+y))-(x-y, A(x-y)) +i(x+iy, A(x+iy))-i(x-iy, A(x-iy))\},$$

it follows that

$$|(x, Ay)|\leqq(1/4)\beta\{||x+y||^2+||x-y||^2+||x+iy||^2+||x-iy||^2\} =\beta(||x||^2+||y||^2)=2\beta.$$

Hence $\alpha\leqq 2\beta$.

Let P denote a field operator having a state with infinite length in its range (from unit vectors) such as $a^+(\boldsymbol{k})$, $a(\boldsymbol{k})$ $a^+a(\boldsymbol{k})$ or $\varphi(\boldsymbol{x})*\rho(\boldsymbol{x})$ defined in §2.

Theorem 2. $\{\tau_2$ closure of $R(\mathfrak{A})\}$ *does not contain* P.

Proof. From the character of P, we can select an orthonormal set $\{u_n\}=\{\prod_k\otimes\varphi(n_k^4)\}$ such that $||Pu_n||\geqq n^4$ and $Pu_n\perp Pu_m$ for $m\neq n$. (We can prove this theorem without $Pu_n\perp Pu_m$ for $m\neq n$.)

Let $f(n)$ denote the function $f(n)=||Pu_n||\geqq n^4$ defined on the set on n. The state $\psi=\sum_{n=1}^{\infty}(u_n/n)$ contains in $\mathfrak{H}$.

Construct $\varphi=\sum_{n=1}^{\infty}(Pu_n/n||Pu_n||)$ contained in $\mathfrak{H}$, then

$$|\langle\varphi, P\psi\rangle|>\textstyle\sum_{n=1}^{\infty}n^4/n^2=\sum_{n=1}^{\infty}n^2=\infty.$$

From the character of P we can understand $|\langle\varphi, P\psi\rangle|=\infty$ naturally. Hence P is not contained in τ_2 closure of $R(\mathfrak{A})$.

Theorem 3. $\{\tau_1$ closure of $R(\mathfrak{A})\}$ *does not contain* P.

Proof. Since P is a field operator, from the proof of Theorem 2 it follows that there exist $\varphi, \psi\in\mathfrak{H}$ with norm 1 such that $|\langle\varphi, P\psi\rangle|=\infty$.

From the proof of Theorem 1, there exist $\Phi\in\mathfrak{H}$ with norm 1 such that $|\langle\Phi, P\Phi\rangle|=\infty$.

Hence P is not contained in $\{\tau_1$ closure of $R(\mathfrak{A})\}$.

§4. The use of the conditional convergence (E. R. Integral).

(1) Let $(\mathfrak{S})$ denote the testing function's space;

$$(\mathfrak{S})=\{\varphi(n);\ \lim_{n\to\infty} n^k|\varphi(n)|=0 \text{ for any } k>0\}.$$

Let's represent $(\psi_1, A\psi_2)$ by the functional defined on $(\mathfrak{S})\times(\mathfrak{S})$. For $\psi_1=\sum_{n(k)}\alpha_{n(k)}\prod_k\otimes\psi_1(n_k)$ and $\psi_2=\sum_{n(k)}\beta_{n(k)}\prod_k\otimes\psi_2(n_k)$, we define

$$\psi_1(\varphi_1)=\sum_{n(k)}(\alpha_{n(k)}\varphi_1(\textstyle\sum_k n(k)))\prod_k\otimes\varphi_1(n_k)$$

and

$$\psi_2(\varphi_2)=\sum_{n(k)}(\beta_{n(k)}\varphi_2(\textstyle\sum_k n(k)))\prod_k\otimes\psi_2(n_k).$$

Here $n(k)$ is the non negative integer valued function of k and φ_1, φ_2 are contained in $(\mathfrak{S})$.

Then, $(\psi_1, A\psi_2)$ is represented by functional

$$[\varphi_1(\psi_1, A\psi_2)\varphi_2]=(\psi_1(\varphi_1), A\psi_2(\varphi_2)).$$

Then suitable observable is A such that this functional can be defined for any pair (ψ_1, ψ_2).

(2) The observable by E. R. Integral. Let's decompose ψ_1 and ψ_2 in $\psi_1=\sum C_i^{(1)}\varphi_i$ and $\psi_2=\sum C_i^{(2)}\varphi_i$, where $\varphi_i=\prod_k\otimes\varphi(n_k^{(i)})$. Furthermore, suppose that $A\psi_2$ can be decomposed in $A\psi_2=\sum C_i^{(A)}\varphi_i$. Then, $(\psi_1, A\psi_2)=\sum_i C_i^{(1)}\overline{C_i^{(A)}}$.

If A satisfies the following conditions, we say that A is contained in the set of extended observables.

(α) For any $\psi_2\in\mathfrak{H}$, $C_i^{(A)}$ are finite and fixed for all i.

(β) $\sum_i C_i^{(1)}\overline{C_i^{(A)}}$ is defined by the following meaning. (From Example 1, finiteness is omitted.) Let's consider the bounded set of k.

Using the function $n(k_i)$ $(i=1,\cdots,n)$, construct the sequence $n^{(1)}(k_i)$, $n^{(2)}(k_i),\cdots$, such that $\sum_{i=1}^n n^{(l)}(k_i)\leqq\sum_{i=1}^n n^{(m)}(k_i)$ for $l<m$. Furthermore, construct the following function $f(x)$:

$$f(x)=2\ C_{n^{(1)}}(k_i) \quad \text{for } \tfrac{1}{2}<x\leqq 1$$
$$f(x)=2^2\ C_{n^{(2)}}(k_i) \quad \text{for } (\tfrac{1}{2})^2<x\leqq\tfrac{1}{2},$$
$$\cdots\cdots\cdots\cdots$$
$$f(x)=2^m C_{n^{(m)}}(k_i) \quad \text{for } (\tfrac{1}{2})^n<x\leqq(\tfrac{1}{2})^{n-1},\cdots.$$

Here $C_{n^{(m)}}(k_i)$ correspond to the above $C_i^{(1)}\overline{C_i^{(A)}}$. $\sum_i C_i^{(1)}\overline{C_i^{(A)}}=\int_1^1 f(x)dx$ by improper E. R. Integral.

For Example, it is easily seen that the operator $a(0)^+ + a(0)$ or $\varphi(x)*\rho(x)$ is the extended observable, when $\rho(x)$ contains in (Z).

Let's show the conditions of E. R. Integrable.

$$V(F,\nu;f)=\{g(x);\ g(x)-f(x)\in V(F,\nu;0)\}.$$

$V(F,\nu;0)$ is the set of step functions $g(x)=p(x)+r(x)$ satisfied the conditions;

(A) $r(x)=0$ in F. (B) $\int_0^1|p(x)|dx<2^{-\nu}$, (C) $\left|\int_0^1 r(x)dx\right|<2^{-\nu}$ where $F\subset[0,1]$.

If we can select the sequence such that

(a) $V(F_1, \nu_1; f_1) \supseteq V(F_2, \nu_2; f_2) \supseteq \cdots$,

(b) $f(x) = \lim_{m\to\infty} f_m(x)$,

(c) $f_{2n} = f_{2n+1}$, $\nu_{2n} < \nu_{2n+1}$,

(d) $k\{\text{mes}\,[0, 1] - F_{n+1}\} \geq \text{mes}\,\{[0, 1] - F_n\}$ for positive integer $k \geq 2$,

and (e) there exists a function $\phi(n)$ with the properties 1) $\phi(n) > 0$, 2) $\lim_{n\to\infty} \phi(n) = 0$, and 3) on $E \subset [0, 1]$ satisfying the condition

$$\int_E x^{-l} dx < \text{mes}\,\{[0, 1] - F_n\}, \quad \int_E |f_n(x)|\, dx \leq \phi(n)$$

for fixed l, then $f(x)$ is E. R. Integrable [4].

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