82. On a Definition of Singular Integral Operators. II

By Hitoshi KUMANO-GO
Department of Mathematics, Osaka University
(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1964)

2. Main theorems. In this part we shall prove that the main theorems relating to singular integral operators in the sense of A. P. Calderón and A. Zygmund [1] holds for ours defined in the part I by using the lemmas proved there.

Theorem 1. *$H\in S(\lambda, T_s)$ defined in Definition 4 in the part I is a bounded operator in L^2_x and*

$$(2.1)\qquad ||Hu||\leqq\{\delta'/(\delta'-\delta)\}^s\Big(\sum_{i=1}^k A_i\Big)||u||,\quad u\in L^2_x,$$

where $A_i=\sup\limits_{R^n\times\mathcal{D}^*(\eta^{(i)},\delta')}|h_i(x,\zeta)|\cdot\sup\limits_{\eta}|\alpha_i(\eta)|$.

Proof. In the representation (1.16) we have for $u\in L^2_x$

$$||a_i^{(\nu)}H_i^{(\nu)}u||\leqq\sup_{R^n\times\mathcal{D}(\eta^{(i)},\delta)}|a_i^{(\nu)}(x)(\eta-\eta^{(i)})^\nu|\cdot\sup_\eta|\alpha_i(\eta)|\cdot||u||.$$

Hence by (1.14) we have $||a_i^{(\nu)}H_i^{(\nu)}u||\leqq(\delta/\delta')^{|\nu|}A_i||u||$, and therefore

$$||Hu||\leqq\sum_{i=1}^k\sum_\nu(\delta/\delta')^{|\nu|}A_i||u||=\{\delta'/(\delta'-\delta)\}^s\Big(\sum_{i=1}^k A_i\Big)||u||.\quad\text{Q.E.D.}$$

Theorem 2. *Let $H\in S(\lambda, T_s)$ and $\Gamma\in T(p)$, $-\infty<p<+\infty$. Then, for any $\sigma_0\geqq0$ we have the representation*

$$(2.2)\qquad \begin{aligned}\Gamma H-H\Gamma&=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}H_\alpha\cdot(x^\alpha\Gamma)+K^{(1)}_{\sigma_0}\\&=-\sum_{1\leqq|\alpha|\leqq l-1}\frac{1}{\alpha!}(x^\alpha\Gamma)\cdot H_\alpha+K^{(2)}_{\sigma_0}\end{aligned}$$

for every $l>\mathrm{Max}\,[\{(4k+n)\tau+p\}/\rho,0]$ with $k=[\sigma_0/(2\rho)+1]$, where $H_\alpha\in S(\lambda, T_s)$ defined by $\sigma(H_\alpha)(x,\eta)=D^\alpha_x\sigma(H)(x,\eta)$ and $K^{(i)}_{\sigma_0}$ $(i=1,2)$ are of order σ_0 such that

$$||\Lambda^{\sigma_1}K^{(i)}_{\sigma_0}\Lambda^{\sigma_2}||\leqq C_{\sigma_0,l,\tau}\Big(\frac{\delta'}{\delta'-\delta}\Big)^s\sum_{i=1}^k\Big\{\sum_{|\beta|\leqq4k+l}\sup_{R^n\times\mathcal{D}^*(\eta^{(i)},\delta')}|D^\beta_xh_i(x,\zeta)|\cdot\sup_\eta|\alpha_i(\eta)|\Big\}.$$

Corollary. *If $H\in S(\lambda, T_s)$ and $\Psi\overset{\infty}{\equiv}0$, then $H\Psi\overset{\infty}{\equiv}\Psi H\overset{\infty}{\equiv}0$.*

Proof. By (1.16) and (1.17) we have $\Gamma H-H\Gamma=\sum_{i=1}^k\sum_\nu(\Gamma a_i^{(\nu)}-a_i^{(\nu)}\Gamma)H_i^{(\nu)}$ and by Lemma 3

$$\Gamma a_i^{(\nu)}-a_i^{(\nu)}\Gamma=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}D^\alpha_xa_i^{(\nu)}\cdot(x^\alpha\Gamma)+K^{(\nu)}_{\sigma_0,i}\equiv I_i^{(\nu)}+K^{(\nu)}_{\sigma_0,i}.$$

It is easy to see

$$\sum_{i=1}^k\sum_\nu I_i^{(\nu)}=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}H_\alpha\cdot(x^\alpha\Gamma).$$

If we estimate the operator norm $\|\Lambda^{\sigma_1}K^{(1)}_{\sigma_0}\Lambda^{\sigma_2}\|\leqq\sum_{i,\nu}\|\Lambda^{\sigma_1}K^{(\nu)}_{\sigma_0,i}\Lambda^{\sigma_2}\|$ by (1.19), we get the first part of (2.2). The second part is obtained from that of (1.18).

Now for $H\in\boldsymbol{S}(\lambda, T_s)$ we define $H^{\#}\in\boldsymbol{S}(\lambda, T_s)$ by $\sigma(H^{\#})(x,\eta)=\overline{\sigma(H)(x,\eta)}$ and for H_1, $H_2\in\boldsymbol{S}(\lambda, T_s)$ define $H_1\circ H_2\in\boldsymbol{S}(\lambda, T_s)$ by $\sigma(H_1\circ H_2)(x,\eta)=\sigma(H_1)(x,\eta)\cdot\sigma(H_2)(x,\eta)$.

Theorem 3. *i)* $H\in\boldsymbol{S}(\lambda, T_s)\Rightarrow H^*\overset{\rho}{\equiv}H^{\#}$, *ii)* H_1, $H_2\in\boldsymbol{S}(\lambda, T_s)\Rightarrow H_1\circ H_2\overset{\rho}{\equiv}H_1H_2\overset{\rho}{\equiv}H_2H_1$, *where H^* means the adjoint operator of H.*

Proof. i) Using (1.16) we can write $H^*=\sum_{i=1}^{k}\sum_{\nu}\overline{H}_i^{(\nu)}\overline{a}_i^{(\nu)}$ where $\overline{H}_i^{(\nu)}$ are defined by $\widehat{\overline{H}_i^{(\nu)}u}(\xi)=\overline{h}_i^{(\nu)}(\eta(\xi))\hat{u}(\xi)$. On the other hand $H^{\#}=\sum_{i=1}^{k}\sum_{\nu}\overline{a}_i^{(\nu)}\overline{H}_i^{(\nu)}$. As $\Gamma\overline{H}_i^{(\nu)}=\overline{H}_i^{(\nu)}\Gamma\in\boldsymbol{T}(p)$ for $\Gamma\in\boldsymbol{T}(p)$, we have by Lemma 3

$$(\Gamma\overline{H}_i^{(\nu)})\overline{a}_i^{(\nu)}-\overline{a}_i^{(\nu)}(\overline{H}_i^{(\nu)}\Gamma)=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}D_x^{\alpha}\overline{a}_i^{(\nu)}\cdot(x^{\alpha}\overline{H}_i^{(\nu)}\Gamma)+K^{(\nu)}_{\sigma_0,i}.$$

We can write

$$\begin{aligned}\mathfrak{F}[x^{\alpha}\overline{H}_i^{(\nu)}\Gamma]&=(\sqrt{-1}\,D_{\xi})^{\alpha}(\overline{h}_i^{(\nu)}(\eta(\xi))\hat{\gamma}(\xi))\\&=\sum_{|\beta|\leqq|\alpha|}C_{\alpha,\beta}D_{\eta}^{\beta}\overline{h}_i^{(\nu)}(\eta(\xi))\cdot\hat{\gamma}_{\alpha,\beta}(\xi)\end{aligned}$$

where $\hat{\gamma}_{\alpha,\beta}(\xi)$ satisfy (1.6) by (1.9), so that

$$\Gamma_{\alpha,\beta}=\gamma_{\alpha,\beta}*\in\boldsymbol{T}(p-\rho|\alpha|)\subset\boldsymbol{T}(p-\rho)\text{ for }|\alpha|\neq 0.$$

Hence we have by (1.19)

$$\Gamma H^*-H^{\#}\Gamma=\sum_{0\leqq|\alpha|\leqq l-1,\ |\beta|\leqq|\alpha|}C'_{\alpha,\beta}H^{\#}_{\alpha,\beta}\Gamma_{\alpha,\beta}+K'_{\sigma_0}\tag{2.3}$$

where $H^{\#}_{\alpha,\beta}\in\boldsymbol{S}(\lambda, T_s)$ defined by $\sigma(H^{\#}_{\alpha,\beta})(x,\eta)=\overline{D_x^{\alpha}D_{\eta}^{\beta}\sigma(H)(x,\eta)}$ and $\Gamma_{\alpha,\beta}\in\boldsymbol{T}(p-\rho)$. Considering $\Gamma(H^*-H^{\#})=(\Gamma H^*-H^{\#}\Gamma)+(H^{\#}\Gamma-\Gamma H^{\#})$, we get from Theorem 2 and (2.3)

$$\Gamma(H^*-H^{\#})=\sum_{j=1}^{l'}H_j\Gamma_j+K_{\sigma_0}\tag{2.4}$$

where $H_j\in\boldsymbol{S}(\lambda, T_s)$ and $\Gamma_j\in\boldsymbol{T}(p-\rho)$.

Let $\Gamma_0\in\boldsymbol{T}(0)$ such that $\Gamma_0=\gamma_0*$ where $\hat{\gamma}_0(\xi)\in C^{\infty}(E_{\xi}^{n})$ such that $\hat{\gamma}_0(\xi)=1$ for $|\xi|\geqq 2$, $=0$ for $|\xi|\leqq 1$. If K^* is of order $\sigma\geqq 0$, K is of order σ. Hence we have by Lemma 2 and Corollary of Theorem 2

$$H^*\Gamma=\Gamma_0H^*\Gamma+(1-\Gamma_0)H^*\Gamma\overset{\infty}{\equiv}\Gamma_0H^*\Gamma.$$

Applying (2.4) to $\Gamma_0(H^*-H^{\#})$ we have $\Gamma_0(H^*-H^{\#})\Gamma=\sum_{j=1}^{l''}H'_j\Gamma'_j+K'_{\sigma_0}$ where $H'_j\in\boldsymbol{S}(\lambda, T_s)$, $\Gamma'_j\in\boldsymbol{T}(p-\rho)$ and K'_{σ_0} are of order σ_0. This completes the proof of i).

ii) Let $H_1=\sum_{i=1}^{k}\sum_{\nu}a_{1,i}^{(\nu)}H_{1,i}^{(\nu)}$ and $H_2=\sum_{i'=1}^{k'}\sum_{\nu'}a_{2,i'}^{(\nu')}H_{2,i'}^{(\nu')}$. Then, $H_1\circ H_2=\sum_{i,i',\nu,\nu'}a_{1,i}^{(\nu)}a_{2,i'}^{(\nu')}H_{1,i}^{(\nu)}H_{2,i'}^{(\nu')}$. We consider

$$\begin{aligned}&\Gamma(a_{1,i}^{(\nu)}H_{1,i}^{(\nu)}a_{2,i'}^{(\nu')})-(a_{1,i}^{(\nu)}a_{2,i'}^{(\nu')})H_{1,i}^{(\nu)}\Gamma\\&=(\Gamma a_{1,i}^{(\nu)}-a_{1,i}^{(\nu)}\Gamma)H_{1,i}^{(\nu)}a_{2,i'}^{(\nu')}+a_{1,i}^{(\nu)}\{(\Gamma H_{1,i}^{(\nu)})a_{2,i'}^{(\nu')}-a_{2,i'}^{(\nu')}(\Gamma H_{1,i}^{(\nu)})\},\end{aligned}$$

and $\Gamma(H_1H_2-H_1\circ H_2)=(\Gamma H_1H_2-H_1\circ H_2\Gamma)+(H_1\circ H_2\Gamma-\Gamma H_1\circ H_2)$. This shows that ii) is proved by the similar way as the proof of i).

Q.E.D.

Theorem 4. *Let $H\in \boldsymbol{S}(\lambda, T_s)$ and $\Gamma\in \boldsymbol{T}(p)$, $0<p\leqq\rho$. Suppose $|\sigma(H)(x,\eta)|\geqq\delta>0$ in $R_x^n\times\Omega$ for an open set $\Omega\subset E_\eta^s$. Then, for any $0<\varepsilon_0<1$ and compact set $\Xi\subset\Omega$, there exists a constant $C_{\varepsilon_0,\Xi}$ such that*

$$\|H\Gamma u\|^2\geqq(1-\varepsilon_0)\delta^2\|\Gamma u\|^2-C_{\varepsilon_0,\Xi}\|u\| \tag{2.5}$$

for every $u\in\mathfrak{H}_p$ such as $\{\eta(\xi);\ \xi\in\operatorname{supp}\hat{u}(\xi)\}\subset\Xi$.

Proof. Let $\{\Theta_g(x)\}$ be the partition of the unity such that

$$\Theta_g(x)\in C_0^\infty(\mathcal{D}_{g,\varepsilon}),\ \sum_g\Theta_g^2(x)=1,\ |D_x^\alpha\Theta_g(x)|\leqq C_\alpha\varepsilon^{-|\alpha|} \tag{2.6}$$

where $g=\varepsilon/\sqrt{n}\ g^0$ for lattice points g^0 in R_x^n (see [3]). We define $H_g\in\boldsymbol{S}(\lambda,T)$ by $\sigma(H_g)(\eta)=\sigma(H)(g,\eta)$ which are the functions independent of x. Take $\varphi(\eta)\in C_0^\infty(\Omega)$ such that $\varphi(\eta)=1$ on Ξ and define a convolution operator Φ by $\widehat{\Phi u}(\xi)=\varphi(\eta(\xi))\hat{u}(\xi)$. Then $\Phi u=u$, if $\{\eta(\xi);\ \xi\in\operatorname{supp}\hat{u}(\xi)\}\subset\Xi$. Using (1.16) we have

$$\begin{aligned}\|H\Gamma u\|^2&=\sum_g\|\sum_{i=1}^k\sum_\nu(\Theta_g a_i^{(\nu)})H_i^{(\nu)}\Gamma\Phi u\|^2\\ &=\sum_g\|\sum_{i,\nu}\Theta_g(x)(a_i^{(\nu)}(x)-a_i^{(\nu)}(g))H_i^{(\nu)}\Gamma\Phi u\\ &\quad+\sum_{i,\nu}\{(\Theta_g(x)a_i^{(\nu)}(g))(H_i^{(\nu)}\Gamma\Phi)-(H_i^{(\nu)}\Gamma\Phi)(\Theta_g(x)a_i^{(\nu)}(g))\}u\\ &\quad+H_g\Phi\Gamma\Theta_g u\|^2\equiv\sum_g\|I_{1,g}+I_{2,g}+H_g\Phi\Gamma\Theta_g u\|^2\\ &\geqq(1-\varepsilon_1)\sum_g\|H_g\Phi\Gamma\Theta_g u\|^2-2\varepsilon_1^{-1}(\sum_g\|I_{1,g}\|^2+\sum_g\|I_{2,g}\|^2).\end{aligned} \tag{2.7}$$

In general, we have for a sequence $\{a_\nu\}$ of complex numbers.

$$\begin{aligned}|\sum_\nu a_\nu|^2&=|\sum_\nu\{(\nu_1+1)^{-1}\cdots(\nu_s+1)^{-1}\}\{(\nu_1+1)\cdots(\nu_s+1)a_\nu\}|^2\\ &\leqq C_s\sum_\nu(\nu_1+1)^2\cdots(\nu_s+1)^2|a_\nu|^2.\end{aligned} \tag{2.8}$$

As $|\Theta_g(x)(a_i^{(\nu)}(x)-a_i^{(\nu)}(g))|\leqq C\varepsilon\operatorname{Max}_{|\alpha|=1}|D_x^\alpha a_i^{(\nu)}|$ and $\Gamma\Phi u=\Gamma u$, we have by (1.14)

$$\begin{aligned}\sum_g\|I_{1,g}\|^2&\leqq C'\varepsilon^2\sum_{i,\nu}(\nu_1+1)^2\cdots(\nu_s+1)^2\operatorname{Max}_{|\alpha|=1}|D_x^\alpha a_i^{(\nu)}|^2\cdot\|H_i^{(\nu)}\Gamma u\|^2\\ &\leqq C_H\varepsilon^2\|\Gamma u\|^2.\end{aligned} \tag{2.9}$$

Applying (2.8) and Lemma 4, we have

$$\sum_g\|I_{2,g}\|^2\leqq C_{H,\varepsilon}\|u\|^2. \tag{2.10}$$

As $\operatorname{supp}\mathfrak{F}[\Phi\Gamma\Theta_g u](\xi)\subset\operatorname{supp}\varphi(\eta)\subset\Omega$, we have by assumption

$$\begin{aligned}\|H_g\Phi\Gamma\Theta_g u\|^2&=\|\sigma(H_g)(\eta(\xi))\widehat{\Phi\Gamma\Theta_g u}(\xi)\|^2\geqq\delta^2\|\Phi\Gamma\Theta_g u\|^2\\ &\geqq\delta^2\{(1-\varepsilon_1)\|\Theta_g\Gamma\Phi u\|^2-\varepsilon_1^{-1}\|\{(\Gamma\Phi)\Theta_g-\Theta_g(\Gamma\Phi)\}u\|^2\}.\end{aligned}$$

Applying Lemma 4 again, we have by (2.6)

$$\sum_g\|H_g\Phi\Gamma\Theta_g u\|^2\geqq(1-\varepsilon_1)\delta^2\|\Gamma u\|^2-C_\varepsilon\varepsilon_1^{-1}\delta^2\|u\|^2. \tag{2.11}$$

If we fix ε_1 such as $(1-\varepsilon_1)^2\geqq(1-\varepsilon_0/2)$ and take ε such as $2\varepsilon_1^{-1}C_H\varepsilon^2\leqq\delta^2\varepsilon_0/2$, we get (2.5).

Q.E.D.

Example. Let $(m, \mathfrak{m})=(m, m_1, \cdots m_n)$ be a real vector whose elements are positive integers.

Setting $|\alpha:\mathfrak{m}|=\alpha_1/m_1+\cdots+\alpha_n/m_n$, we consider a differential operator of the form

$$L(t, x, D_t, D_x)=\sum_{i/m+|\alpha:\mathfrak{m}|=1} a_{i,\alpha}(t, x)D_t D_x \quad (a_{m,\alpha}(t, x)=1)$$

where $a_{i,\alpha}(t, x)\in\mathscr{B}_x$ for fixed $t\in[0, T]$, $T>0$, and are sufficiently differentiable with respect to t in the topology of $\mathscr{B}_x$.

Set $\hat{\lambda}(\xi)=\left\{\sum_{j=1}^n \xi_j^{2m_j}\right\}^{1/2m}$. Then, $\hat{\lambda}(\xi)$ satisfies (1.4) for $\rho=\operatorname{Min}_{1\leqq j\leqq n} m/m_j$ and $\tau=\operatorname{Max}_{1\leqq j\leqq n} m/m_j$. The transformation $T_n(s=n)$ is defined by $T_n:\eta_j(\xi)=\xi_j\hat{\lambda}(\xi)^{-m/m_j}$, $j=1,\cdots, n$. Then T_n satisfies (1.9), and for some positive constants $C_1<C_2$, we have

$\Xi=$the closure of $\{\eta(\xi)=(\eta_1(\xi),\cdots,\eta_n(\xi));\ \xi\in E_\xi^\eta\}\subset\{\eta;\ C_1\leqq|\eta|\leqq C_2\}$.

If we define $H^{(i)}(t)\in\boldsymbol{S}(\lambda, T_n)$ $(i=0,\cdots, m)$ with t as a parameter by $\sigma(H^{(i)}(t))(x,\eta)=\sum_{i/m+|\alpha:\mathfrak{m}|=1} a_{i,\alpha}(t, x)(\sqrt{-1}\,\eta)^\alpha\alpha_0(\eta)$ where $\alpha_0(\eta)\in C_0^\infty(E_\eta^n)$ such that $\alpha_0(\eta)=1$ on Ξ, we have formally for functions $u(t, x)$

$$L(t, x, D_t, D_x)u=\frac{1}{\sqrt{2\pi}^n}\int e^{\sqrt{-1}\,x\cdot\xi}L(t, x, D_t, \sqrt{-1}\,\xi)\hat{u}(t,\xi)\,d\xi$$

$$=\sum_{i=0}^m H^{(i)}(t)\Lambda^{m-i}D_t^i u.$$

Furthermore we assume that $L(t, x, \lambda, \sqrt{-1}\,\zeta)$ is resolve into

$$L(t, x, \lambda, \sqrt{-1}\,\zeta)=\prod_{j=1}^m(\lambda+\lambda_j(t, x, \zeta))$$

where $\lambda_j(t, x, \zeta)$ are analytic in some complex neighborhood Ω^* of Ξ. Then we can take $\mathscr{D}(\eta^{(i)},\delta)\subset\mathscr{D}^*(\eta^{(i)},\delta')$ $(i=1,\cdots,k)$ such that $\Xi\subset\bigcup_{i=1}^k\mathscr{D}(\eta^{(i)},\delta)$, $\bigcup_{i=1}^k\mathscr{D}^*(\eta^{(i)},\delta')\subset\Omega^*$, and $\alpha_i(\eta)\in C_0^\infty(\mathscr{D}(\eta^{(i)},\delta))$ such that $\sum_{i=1}^k\alpha_i(\eta)=1$ on Ξ.

Now we consider $H_j(t)\in\boldsymbol{S}(\lambda, T_n)$ defined by $\sigma(H_j(t))(x,\eta)=\sum_{i=1}^k\lambda_j(t, x,\eta)\alpha_i(\eta)$ and set $L_0=\prod_{j=1}^m(D_t+H_j(t)\Lambda)$. Then we can prove

$$\|(L-L_0)u\|^2\leqq C\sum_{0\leqq i+\sigma\leqq m-\rho_0}\|D_t^i\Lambda^\sigma u\|,\ \rho_0=\operatorname{Min}[\rho, 1]$$

by Theorems 1 and 2, if we consider $\Phi\Lambda\in\boldsymbol{T}(1)$ instead of Λ where Φ is an operator of $\boldsymbol{T}(0)$ defined in the proof of Theorem 4 (see [4]). The example of T_s where $s>n$ will be published later.

References

[1] A. P. Calderón and A. Zygmund: Singular integral operators and differential equations. Amer. J. Math., **79**, 901-921 (1957).

[2] A. P. Calderón: Uniqueness in the Cauchy problem for partial differential equations. Amer. J. Math., **80**, 16-36 (1958).

[3] L. Hörmander: On the uniqueness of the Cauchy problem II. Math. Scand., **7**, 177–190 (1959).
[4] H. Kumano-go: On the uniqueness for the solution of the Cauchy problem. Osaka Math. J., **15**, 151–172 (1963).
[5] ——: On the uniqueness of solutions of the Cauchy problem for hypoelliptic partial differential operators. Proc. Japan Acad., **39**, 342–347 (1963).
[6] M. Matsumura: Existence locale de solutions pour quelques systemes d'équations aux dérivées partielles. Jap. J. Math., **32**, 13–49 (1962).
[7] S. Mizohata: Unicité du prolongement des solutions des équations elliptiques du quatrième ordre. Proc. Japan Acad., **34**, 687–692 (1958).
[8] ——: Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboriques. Mem. Coll. Sci. Univ. Kyoto, Ser. A, **31**, 219–239 (1958).
[9] ——: Le problème de Cauchy pour le passé pour quelques équations paraboliques. Proc. Japan Acad., **34**, 693–696 (1958).
[10] ——: Systèmes hyperboliques. J. Math. Soc. Japan, **11**, 205–233 (1959).
[11] L. Schwarz: Théorie des distribution, I, II. Paris, Hermann (1950/50).
[12] M. Yamaguti: Le problème de Cauchy et les opérateurs d'intégrale singulière. Mem. Coll. Sci. Kyoto Univ. Ser. A, **32**, 121–151 (1959).