110. *On the Definition of Functional Integrals*

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§1. **Introduction.** Functional integral is one of the powerful tool in quantum field theory or stocastic process [1–2]. The trial to define it exactly and naturally has been done by K. O. Friedricks [1]. But his definition is still restricted by the usual Hilbert space $\mathfrak{H}$. Many definitions of it are given, but each of them is not sufficient and we must show the more suitable new definition, because they cannot describe the important integral skilfully. Therefore, the integral of functions defined in the Hilbert space (or in its extension) is not necessarily used in usual, though it is one of the powerful tools.

Here, let's investigate precisely the definition in [1–2] and give the generalized definition which is faithful to the following Example 1. Here, Example 1 is the most basic one showing the natural explanation of this integral. Our method is one corresponding to the continuous representation of states [5–6].

On the other hand, Feynman integral (a sort of singular functional integral) is one of the important purpose of this research. We have already succeed to define it by using E. R. Integral which is the most general singular integral. For functional, this mild integral is effective specially [7], [1].

In the next paper, we will show it. In this paper, we give the relation between Feynman integral and our generalized definition for preliminalies.

§2. **Definition of the functional integral.** Let $\mathfrak{H}$ denote the space of the real square integrable functions defined on the real axis. (We may change to complex valued functions easily.)

Let $f(\xi(s))$ (for $\xi(s)$, $-\infty < s < +\infty$) denote the real valued functional defined on $\mathfrak{H}$ and $I[f(\xi(s))]$ denote the integral of $f(\xi(s))$, namely $I[f(\xi(s))] = \int_{\mathfrak{H}} f(\xi(s))\, dm(\xi(s))$.

Here we show the most elementary example of the functional integral, and give the generalized definition which is faithful to this example.

Example 1. Let $\mathfrak{H}$ denote the space of sequences $\{x_1, x_2, \cdots\}$ with the property $\Sigma_{i=1}^{\infty} |x_i|^2 < +\infty$. (For example, the sequence of Fourier coefficients for some fixed base.)

$$I[f(x_1, x_2, \cdots)] = \lim_{n\to\infty} \int \cdots \int (1/\sqrt{2\pi}\,\sigma)^n \exp\{-\Sigma_{i=1}^{n} x_i^2/2\sigma^2\} \times$$

$$\times f(x_1, x_2, \cdots, x_n, 0, \cdots) dx_1 \cdots dx_n. \tag{1}$$

The definition in [1] is same as this Example 1 in essential point.

From now let's show the essential point of Example 1 and compair with the definition in [1]. Let $\boldsymbol{x}$ denote the infinite demensional vector $\{x_1, x_2, \cdots, x_n, \cdots\}$ with finite norm and $P(R^n)$ denote the following projection operator to the subspace $R^n \subset \mathfrak{H}$:

$$P(R^n)\boldsymbol{x} = \{x_1, x_2, \cdots, x_n, 0, \cdots\}.$$

Let $\boldsymbol{e}_n$ denote the element of the base $\{0, \cdots, 0, 1, 0, \cdots\}$ in which only the n-th component is 1. $P(R^n)\boldsymbol{x} = \Sigma_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_n \rangle \boldsymbol{e}_n$. The essential tool in Example 1 is this $P(R^n)$. In [1] we use the following projection corresponding to this. Divide the real axis by finite points $\{s_i^{(n)}, i = 0, 1, \cdots, n\}$ and construct the following orthonormal step function $\varphi_i^{(n)}$ in L^2;

$$\varphi_i^{(n)}(s) = \begin{cases} \sqrt{1/(s_i^{(n)} - s_{i-1}^{(n)})} & \text{for } s \in (s_{i-1}^{(n)}, s_i^{(n)}) \\ 0 & \text{for the other } s \ (i = 1, \cdots, n), \end{cases}$$

then $P^{(n)}\xi(s) = \Sigma_{i=1}^n \langle \xi(s), \varphi_i^{(n)}(s) \rangle \varphi_i^{(n)}(s)$.

$$\begin{aligned} I[f(\xi(s))] &= \lim_{D_n \to \infty} \int \cdots \int (1/\sqrt{2\pi}\,\sigma)^n \exp\{-\Sigma_{i=1}^n \langle \xi(s), \varphi_i^{(n)}(s) \rangle^2 / 2\sigma^2\} \times \\ &\quad \times f(p^{(n)}\xi(s)) \ \textstyle\prod_{i=1}^n d\langle \xi(s), \varphi_i^{(n)}(s) \rangle \\ &= \lim_{D_n \to \infty} \int \cdots \int (1/\sqrt{2\pi}\,\sigma)^n \exp\{-\Sigma_{i=1}^n \langle \xi(s), \varphi_i^{(n)}(s) \rangle^2 / 2\sigma^2\} \times \\ &\quad \times \tilde{f}(\langle \xi(s), \varphi_i(s) \rangle \ i = 1, 2, \cdots, n) \textstyle\prod_{i=1}^n d\langle \xi(s), \varphi_i^{(n)}(s) \rangle \\ &\equiv \lim_{D_n \to \infty} I_{n\sigma}[f(\xi(s))]. \end{aligned} \tag{2}$$

If σ tends to ∞, then $\lim_{\sigma \to \infty} \lim_{D_n \to \infty} (\sqrt{2\pi}\,\sigma)^n I_{n\sigma}[f(\xi(s))]$ tends to the functional integral with respect to the ordinary measure. Since Gauss measure is compatible to the increasing of the divided points, we use it at the first step. Now we must give the exact meaning of $\lim_{D_n \to \infty}$.

Let $D = \{D^{(n)}\}$ denote the family of the set of divided points. By the order $D^{(n)} \subseteq D^{(m)}$ which means the inclusion, D becomes to a partially ordered set. Let $\mathfrak{H}_{[s_1, s_2)}$ denote the Hilbert space whose base is $\{\sqrt{\delta(s)}; s_1 \leqq s < s_2\}$. Here the element of this base $\sqrt{\delta(s)}$ has been already defined by using the sequence space [4], [6].

Definition 1. Ordered family is the family of the projection's set $[\{P_j^{(n)}\}]$ with the following properties, where n describes the set belonging to it and j describes the projection elements belonging to this set;

(1) for any k, and for any n, m with the property $n < m$
$P_j^{(n)} \geqq P_k^{(m)}$ or $P_j^{(n)} \perp P_k^{(m)}$ for all j,

(2) $P_j^{(n)} \perp P_i^{(n)}$ for $i \neq j$,

(3) $\vee_j P_j^{(n)} \leqq \vee_k P_k^{(m)}$ for $m \geqq n$,

(4) $\vee_n (\vee_j P_j^{(n)}) = 1$.

Next, let's define the characteristic vector $V(P)$ and characteristic projection $P(\xi)$ corresponding to the above $\varphi_i^{(n)}(s)$ and $\langle \xi(s), \varphi_i^{(n)}(s)\rangle$.

Definition 2. $V(P; \{e_n\})$ is an unit vector effected equally by the elements of some fixed base $\{e_n\}$ of the range of P. $P_{\{e_n\}}(\xi)$ is the formal inner product $\langle \xi, V(P; \{e_n\})\rangle$.

Since $V(P; \{e_n\})$ is not necessarily contained in the original Hilbert space $\mathfrak{H}$, we use the term "formal inner product".

Now, let's use the abbreviation $V(P_j^{(n)}; \sqrt{\delta(s)}, s \in [s_{j-1}^{(n)}, s_j^{(n)})) \equiv V(P_j^{(n)})$, and $P_j^{(n)}(\xi) \equiv \langle \xi, V(P_j^{(n)})\rangle$. We can easily show that $P_j^{(n)}(\xi) V(P_j^{(n)}) \equiv \varphi(s; n, j, \xi) = \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \xi(s)ds/(s_j^{(n)} - s_{j-1}^{(n)})$ in $s \in [s_{j-1}^{(n)}, s_j^{(n)})$. The exact definition of $V(P)$ and $P(\xi)$ by using the discussion in [4] is shown in §4.

Lemma 1. *Suppose that $P_j (j=1,\cdots,n)$ have the properties*

1) $P = \vee_{j=1}^n P_j$, *and* 2) $P_i \perp P_j$ *for* $i \neq j$.

Then $V(P) = \Sigma_{k=1}^n \sqrt{\dim P_k/\dim P}\, V(P_n)$ *and* $P(\xi) = \Sigma_{k=1}^n \sqrt{\dim P_k/\dim P}\, P_k(\xi)$.

This lemma corresponds to the additivity of the integral's domain.

Now, we give the definition of the cylinder functional for $P_j^{(n)}(\xi)$.

Definition 3. If there exists N such that

$$f(\Sigma_j \varphi(s; n, j, \xi)) = \tilde{f}(P_j^{(n)}(\xi(s))) = \tilde{f}(P_i^{(m)}(\xi(s))) = f(\Sigma_i \varphi(s; m, i, \xi))$$

for any $m, n > N$, we call f is the cylinder functional. Furthermore we say that this $\{P_j^{(N)}\}$ or D_N characterizes this cylinder functional f.

Cylinder functional can be represented by the function $\tilde{f}_{C,N}(P_j^{(N)}(\xi(s))$ $j=1,\cdots,N)$ defined in finite dimensional Euclidean space, because $[\{P_j^{(n)}\}]$ has the following properties:

$P_k^{(n)} P_j^{(N)} = P_j^{(N)} P_k^{(n)} = P_k^{(n)}$ or 0 and $(\Sigma_k P_j^{(N)} P_k^{(n)})(\xi) = (\Sigma_{\{k;\, P_j^{(N)} P_k^{(n)} \neq 0\}} P_k^{(n)})(\xi) = P_j^{(N)}(\xi)$ for $n \geqq N$.

We can show that the cylinder functional is a sort of step functional and the increasing of the set D_N characterized this cylinder functional corresponds to the refinement of the steps. (Similar discussion contained discrete $\sqrt{\delta}$ can be possible.) These concepts are important in the next paper.

By using the ordered family of projection, we can also show the generalized definition of cylinder functional which contains the special case defined in Example 1.

Theorem 1. *If $f_k(\xi(s))$ $k=1,2,\cdots,n$ are cylinder functionals, then $F(f_k(\xi(s)); k=1,2,\cdots,n)$ is also a cylinder functional, where $F(x_k;\ k=1,2,\cdots,n)$ is a continuous function of x_k.*

Proof. Suppose that the divisions characterized by cylinder functionals $f_k(\xi(s))$ $(k=1,\cdots,n)$ are denoted by D_k $(k=1,\cdots,n)$ and the set of projections related to D_k is denoted by $\{P_j(k)\}$. Denote the set of division $U_{k=1}^n D_k$ by D_Σ and the set of projections related to D_Σ by

$\{P_j(\Sigma)\}$. Then $f_k(\Sigma_i\varphi(s;\Sigma,i,\xi))=f_k(\Sigma_j\varphi(s;k,j,\xi))$ for $k=1,2,\cdots,n$.

If D_m is a refinement of D_Σ and $\{P_j(m)\}$ is the set of projections related to D_m, then $f_k(\Sigma_j\varphi(s;m,j,\xi))=f_k(\Sigma_i\varphi(s;\Sigma,i,\xi))$ for $k=1,2,\cdots,n$. Hence, $F(f_k(\Sigma_j\varphi(s;m,j,\xi))=F(f_k(\Sigma_i\varphi(s;\Sigma,i,\xi)))$. From the above argument $F(f_k(\xi(s)))$ is also a cylinder functional.

For a cylinder functional $f(\xi(s))$, $|f(\xi(s))|$ is also a cylinder functional. Consequently we can define a metric in the set of cylinder functionals C by dist $(f_1,f_2)=I(|f_1-f_2|)$ for $f_1,f_2\in C$. From the definition of functional integral, we can easily obtain the following

Theorem 2. *The space C with the above distance is a metric space.*

The set of integrable functionals is the completion of C using this norm.

Theorem 3. *The functional contained in the set of integrable functionals $\overline{C}$ has one valued functional integral.*

Proof. For f contained in $\overline{C}$, we can choose a Cauchy sequence of the cylinder functionals $\{f_n\}$ with the following property; for any $\varepsilon>0$, there exists N such that $I(|f_n-f_m|)<\varepsilon$ for $m, n>N$. Since for any above Cauchy sequence $\{f_n\}$

$$|I(f_n)-I(f_m)|\leqq|I(f_n-f_m)|\leqq I(|f_n-f_m|)<\varepsilon \tag{3}$$

$\{I(f_n)\}$ is also a Cauchy sequence. By using the above inequality (3), $\lim_{n\to\infty} I(f_n)$ take the same value for any above equivalent sequence $\{f_n\}$. Using this Theorem 3, we can understand the meanings of $\lim_{\Sigma n\to\infty}$.

§3. **Feynman integral.** The transition probability from the state Ψ at time t_0 to the state ϕ at t can be represented by

$$K(\phi,t;\Psi,t_0)=\int dq'dq''\psi^*(q'')K(q'',t;q',t_0)\phi(q').$$

Here, the Kernel function $K(q''t;q't_0)$ is represented by

$$K(q''t;q't_0)=\lim_{n\to\infty}\int\cdots\int\prod_{k=1}^{n-1}dq_k\prod_{k=0}^{n-1}K(q_{k+1}t_{q+1};q_kt_k).$$

Furthermore $\prod_{k=0}^{n-1}K_k$ is represented by the formula $N_k\cdot e^{(i/h)\Sigma_{k=0}^{n-1}s_k}$ with the normalization factor N_k in which $\Sigma_{k=0}^{n-1}s_k$ tends to

$$\int_{t_0}^{t}dtL[\dot q(t),q(t)]=\int_{t_0}^{t}(m\dot q(s)^2/2-V[q(s)])ds.$$

This formula can be related to Schrödinger equation $\partial u/\partial t=i[(1/(2m))\Delta-V]u(t)$ with the initial condition $u(0)=\psi$, where Δ is the Laplace operator $\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_l^2$ on l-dimensional Euclidean space R^l. Its kernel function is $U_{m,V}^t=\exp it[(1/(2m))\Delta-V]$. For $1/(2m)=0$, $U_{m,V}^t$ becomes to $M_V^t=\exp(-itV)$ and for $V=0$ $U_{m,V}^t$ becomes to

$$K_m^t\psi(x)=(2\pi it/m)^{-l/2}\int\exp[(im/2)(|x-y|^2/t)]\psi(y)dy.$$

Trother's theorem asserts that for all ψ in L^2, $U_{m,V}^t\psi=\lim_{n\to\infty}(K_m^{t/n}M_V^{t/n})^n\psi$.

Here, $(K_m^{t/n}M_V^{t/n})^n\psi(x)$ is represented by

$$(2\pi it/nm)^{-ln/2}\int\cdots\int e^{is(x_0,\cdots,x_n;t)}\psi(x)dx_1\cdots dx_n,\ \text{where } x_0=x.$$

$S(x_0,\cdots,x_n;t)$ must be represented by simple and exact form. The following form is one of them;

$$S(x_0,\cdots,x_n;t)=\Sigma_{j=1}^n[(m/2)|x_j-x_{j-1}|^2/(t/n)^2-V(x_j)]t/n.$$

By using this formula, our above integral can be represented by the functional integral

$$U_{m,V}^t\psi=[\lim_{n\to\infty}(2\pi it/nm)^{-ln/2}]\int_{\Omega_x}\exp i\int_0^t[m\omega^2/2-V(\omega(s))]ds\psi(\omega(0))D\omega.$$

Our problems with respect to $S(x_0,\cdots,x_n;t)$ are the following:

(1) to obtain the more simple and more exact approximate formula,

(2) to investigate whether the above approximate formula tends to the functional integral defined by some meaning or not. From the above two view point the discussion in [2] is not necessarily safficient. Then we need the other arguments.

Our definition correspond to use the step function in the domain of $V(\omega(s))$ and to use a step function constructed by a sort of finite difference of the above step function in $m\dot{\omega}^2/2$. This is also another natural consideration. Another purpose of Feynman integral is to obtain the classical path which takes the minimal value of the integral $\int_{t_0}^t dtL[\dot{q}(t),q(t)]$. This classical path is deduced from the solution of the equation $\delta\int_{t_0}^t dtL[\dot{q}(t),q(t)]=0$, namely it is the solution of the equation $-m\dot{q}(s)-V'[q(s)]=0$ by distribution's meaning.

If we calculate Feynman integral by our method, the path with the maximal probability is still this solution.

Namely, we obtain the following

Theorem 4. *The optimal path of* $\int_{t_0}^t dtL[\dot{q}(t),q(t)]$ *is the maximal probable path of* $\int dq\exp\left\{(i/h)\int_{t_0}^t L[\dot{q}(t),q(t)]dt\right\}$.

If $(1/h)\int_{t_0}^t L[\dot{q}(t),q(t)]dt$ tends to $\pm\infty$, then $\int dq\exp\left\{(i/h)\int_{t_0}^t L[\dot{q}(t),q(t)]dt\right\}$ tends to zero. Using the normalization factor N, δ function is obtained as h tend to 0.

§4. $\sqrt{\delta}$ axis. Let's define the real axis in which generalized functions $\sqrt{\delta(t)}$ $(-\infty<t<+\infty)$ are distributed. We denote by $\tilde{\mathfrak{H}}$ the set of the four vectors of convex continuous functions $[\{f_+,f_-,g_+,g_-\}]$ defined in $(-\infty,+\infty)$ the elements of which satisfy the conditions $|f'_+(\infty)-f'_+(-\infty)|<+\infty$, $|f'_-(\infty)-f'_-(-\infty)|<+\infty$ $|g'_+(\infty)-g'_+(-\infty)|<+\infty$ and $|g'_-(\infty)-g'_-(-\infty)|<+\infty$. According to the properties of the convex continuous functions, the set $[\{\sqrt{f''_+}, \sqrt{f''_-}, \sqrt{g''_+}, \sqrt{g''_-}\}]$ can be defined and considered as the set of the sequence of C^∞ functions $\{\{\sqrt{(f^*_+\rho_n)''}\}, \{\sqrt{(f^*_-\rho_n)''}\}, \{\sqrt{(g^*_+\rho_n)''}\}, \{\sqrt{(g^*_-\rho_n)''}\}\}$. It constructs the generalized L^2 space. Using these generalized L^2 space, let's define the characteristic vector $V(P)$ and the characteristic projection $P(\xi)$. Characteristic vector $V(P)$ corresponds to the generalized characteristic function of t produced by the range of P. The reason by which we use the term "generalized characteristic function" is the following, if the interval corresponding to P is one point $t=t_0$, then $V(P)$ becomes to $\sqrt{\delta(t_0)}$. Characteristic projection $P(\xi)$ is defined by $P(\xi(s))=\langle V(P), \xi(s)\rangle$. This $\sqrt{\delta(x)}$ which is the base of our discussion is already shown in [4], and is assemble to one in [6]. The normalization factor by Feynman is

$$N(t, 0)=\lim_{n\to\infty}(m/ih\tau)^{n/2}=\lim_{n\to\infty}(mn/iht)^{n/2}.$$

This infinity is canceled by the functional integral defined in §3 and constructed generalized δ function.

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