108. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XIII

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1964)

Preliminaries. Let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed bounded infinite sequence of complex numbers (counted according to the respective multiplicities), and let $D_1, D_2,\cdots, D_n$ be arbitrarily prescribed, bounded, connected, and closed sets with (linear or planar) positive measures in the complex plane such that they are mutually disjoint and each of them does not contain any point belonging to the closure of $\{\lambda_\nu\}$. Then there are infinitely many bounded normal operators N such that the point spectrum and the continuous spectrum of each N of them are given respectively by $\{\lambda_\nu\}$ and the union of one, D_j, of $D_1, D_2,\cdots, D_n$ and the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself, as can be found from Theorem 29. If we suppose that $\{\varphi_\nu^{(j)}\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu^{(j)}\}_{\mu=1,2,3,\dots}$ are arbitrarily given incomplete orthonormal sets orthogonal to each other such that the complex abstract Hilbert space $\mathfrak{H}$ under consideration is determined by themselves, then one of those bounded normal operators in $\mathfrak{H}$, which will be denoted by N_j, is expressible in the form of

$$N_j=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu^{(j)}\otimes L_{\varphi_\nu^{(j)}}+\sum_{\mu=1}^{\infty}\Psi_\mu^{(j)}\otimes L_{\psi_\mu^{(j)}},$$

where $\Psi_\mu^{(j)}=\sum_{\kappa=1}^{\infty}(N_j\psi_\mu^{(j)}, \psi_\kappa^{(j)})\psi_\kappa^{(j)}$; and moreover, if we denote $(N_j\psi_\mu^{(j)}, \psi_\kappa^{(j)})$ by $\beta_{\mu\kappa}$ for brevity of expression, the matrix-operator $(\beta_{\mu\kappa})$ associated with the infinite matrix where $\beta_{\mu\kappa}$ is the element appearing in row μ column κ is a bounded normal operator with $\sum_{\kappa=1}^{\infty}|\beta_{\mu\kappa}|^2>|\beta_{\mu\mu}|^2>0$ $(\mu=1,2,3,\cdots)$, as we have already demonstrated in the preceding paper. Let now $\{K^{(j)}(\lambda)\}$ be the complex spectral family of N_j; let $\{\psi_{\mu_p}^{(j)}\}_{p=1,2,3,\dots}(\subset\{\psi_\mu^{(j)}\}_{\mu=1,2,3,\dots})$ be the incomplete orthonormal set determining the subspace $K^{(j)}(D_j)\mathfrak{H}$; let $\mathfrak{M}_1$ be the subspace determined by $\{\varphi_\nu^{(1)}\}_{\nu=1,2,3,\dots}$; let $\mathfrak{N}_j$ and $\mathfrak{N}_j'$ be the subspaces determined by $\{\psi_\mu^{(j)}\}_{\mu=1,2,3,\dots}$ and $\{\psi_{\mu_p}^{(j)}\}_{p=1,2,3,\dots}$ respectively; let $f_{1\alpha}$ and $f_{1\alpha}'$ $(\alpha=1,2,3,\cdots,m)$ be arbitrary elements consisting of all $\varphi_\nu^{(1)}$ in $\mathfrak{M}_1$; let $f_{2\alpha}$ and $f'_{2\alpha}$ be arbitrary elements consisting of all $\psi_\mu^{(1)}$ in $\mathfrak{N}_1$; let $g_{j\beta}$ and $g_{j\beta}'$ $(j=1,2,3,\cdots,n;$ $\beta=1,2,3,\cdots,k_j)$ be arbitrary elements consisting of all $\psi_{\mu_p}^{(j)}$ in $\mathfrak{N}_j'$; and let

$$\chi(\lambda)=\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}),\ (f_{1\alpha}'+f_{2\alpha}'))+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g_{j\beta}'),$$

where $1 \leqq m, n, k_j < \infty$.

If we consider a function $T(\lambda)$ such that $T(\lambda)-\chi(\lambda)$ is regular in the domain $\mathfrak{D}(\lambda:|\lambda|<\infty)$, then $T(\lambda)$ has the three following properties.

(i) $T(\lambda)$ is regular in $\mathfrak{D}$ with the exception of the union of the set $\bigcup_{j=1}^{n} D_j$ and the closure of $\{\lambda_\nu\}$, and every point belonging to that union is a singularity of $T(\lambda)$.

(ii) $T(\lambda)$ does not comprize any term with isolated essential singularity in $\mathfrak{D}$, though it may occur that $\lambda=\infty$ is its unique isolated essential singularity.

(iii) If we denote by $K_\nu^{(1)}$ the projector associated with the eigenspace of N_1 corresponding to the eigenvalue λ_ν and put $(K_\nu^{(1)} f_{1\alpha}, f'_{1\alpha})=c_\alpha^{(\nu)}$, then the principal part of $T(\lambda)$ at the pole λ_ν in the sense of the functional analysis is expressed in the form $\sum_{\alpha=1}^{m} c_\alpha^{(\nu)}/(\lambda-\lambda_\nu)^\alpha$ where $\sum_{\nu=1}^{\infty} c_\alpha^{(\nu)}$ is absolutely convergent for $\alpha=1,2,3,\cdots,m$, because of the fact that

$$\begin{aligned}\sum_{\nu=1}^{\infty}|c_\alpha^{(\nu)}| &\leqq \sum_{\nu=1}^{\infty}|(f_{1\alpha}, \varphi_\nu^{(1)})|\,|(\varphi_\nu^{(1)}, f'_{1\alpha})| \\ &\leqq \{\sum_{\nu=1}^{\infty}|(f_{1\alpha}, \varphi_\nu^{(1)}|^2\}^{\frac{1}{2}}\{\sum_{\nu=1}^{\infty}|(\varphi_\nu^{(1)}, f'_{1\alpha})|^2\}^{\frac{1}{2}}=\|f_{1\alpha}\|\,\|f'_{1\alpha}\|<\infty\end{aligned}$$

for $\alpha=1,2,3,\cdots,m$.

We here note that

$$((\lambda I-N_1)^{-\alpha} f_{1\alpha}, f'_{2\alpha})=((\lambda I-N_1)^{-\alpha} f_{2\alpha}, f'_{1\alpha})=0 \quad (\alpha=1,2,3,\cdots,m),$$

as can be verified by making use of the complex spectral family of N_1.

Definitions. The functions $T(\lambda)-\chi(\lambda)$, $\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha} f_{1\alpha}, f'_{1\alpha})=\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty} c_\alpha^{(\nu)}/(\lambda-\lambda_\nu)^\alpha$, and $\chi(\lambda)-\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty} c_\alpha^{(\nu)}/(\lambda-\lambda_\nu)^\alpha=\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha} f_{2\alpha}, f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta} g_{j\beta}, g'_{j\beta})$ are called the ordinary part, the first principal part, and the second principal part of $T(\lambda)$ respectively, as we defined for the function $S(\lambda)$ in Theorem 1 [cf. Proc. Japan Acad., Vol. 38, No. 6, 263–268 (1962)].

Since the incomplete orthonormal sets $\{\varphi_\nu^{(j)}\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu^{(j)}\}_{\mu=1,2,3,\cdots}$ subject to the two conditions stated before are arbitrary, and since the elements $f_{1\alpha}, f'_{1\alpha}\in\mathfrak{M}_1$, $f_{2\alpha}, f'_{2\alpha}\in\mathfrak{N}_1$, and $g_{j\beta}, g'_{j\beta}\in\mathfrak{N}'_j$ are arbitrary as far as the previously given conditions on these elements are satisfied respectively, for any given ordinary part there exist infinitely many functions $T(\lambda)$ each of which is associated with such bounded normal operators as above and satisfies conditions (i), (ii), and (iii) for the fixed sets $\{\lambda_\nu\}$, $D_1, D_2,\cdots, D_n$ and the given positive integers $m, k_1, k_2, \cdots, k_n$. We denote by $\mathfrak{F}$ the class of all these functions $T(\lambda)$.

In this paper we shall discuss the fundamental properties of $T(\lambda)\in\mathfrak{F}$ alone.

Theorem 30. Let $T(\lambda)$ be a function belonging to $\mathfrak{F}$, $R(\lambda)$ the ordinary part of $T(\lambda)$, $\chi(\lambda)$ the sum of the first and the second principal parts of $T(\lambda)$, and Γ a rectifiable closed Jordan curve, positively oriented, such that both the set $\bigcup_{j=1}^{n} D_j$ and the closure of $\{\lambda_\nu\}$ are wholly contained inside Γ itself. Then

$$(30)\quad \frac{1}{2\pi i}\int_\Gamma T(\lambda)(\lambda-z)^{-k}d\lambda=\begin{cases} R^{(k-1)}(z)/(k-1)! \\ \qquad\text{(for every point } z \text{ inside } \Gamma) \\ -\chi^{(k-1)}(z)/(k-1)! \\ \qquad\text{(for every point } z \text{ outside } \Gamma), \end{cases}$$

where $i=\sqrt{-1}$, $k=1,2,3,\cdots$, $0!=1$, and $R^{(0)}(z)=R(z)$.

Proof. By the hypotheses on $T(\lambda)$ and $\chi(\lambda)$, $\chi(\lambda)$ is expressed in the form of

$$\chi(\lambda)=\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}),\ (f'_{1\alpha}+f'_{2\alpha}))+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta})$$

for appropriately chosen $f_{1\alpha}, f_{2\alpha}, f'_{1\alpha}, f'_{2\alpha}, g_{j\beta}, g'_{j\beta}$, and N_j subject to such conditions as were stated before respectively. Since, moreover, $T(\lambda)=R(\lambda)+\chi(\lambda)$, we can easily verify by means of the lemma shown before as a preliminary for Theorem 1 [cf. loc. cit., 263–265 (1962)] and of Cauchy's integral formula that

$$\frac{1}{2\pi i}\int_\Gamma T(\lambda)(\lambda-z)^{-k}d\lambda=\frac{1}{2\pi i}\int_\Gamma R(\lambda)(\lambda-z)^{-k}d\lambda$$
$$=R^{(k-1)}(z)/(k-1)!\quad (k=1,2,3,\cdots)$$

for every point z inside Γ. As can be found similarly from the lemma quoted above and Cauchy's integral theorem, we also obtain

$$\frac{1}{2\pi i}\int_\Gamma T(\lambda)(\lambda-z)^{-k}d\lambda=\frac{1}{2\pi i}\int_\Gamma \chi(\lambda)(\lambda-z)^{-k}d\lambda$$
$$=-\chi^{(k-1)}(z)/(k-1)!\quad (k=1,2,3,\cdots)$$

for every point z outside Γ.

Theorem 31. Let $\chi(\lambda)$ and Γ be the same notations as those in Theorem 30 respectively, and let N be an arbitrary bounded normal operator such that its point spectrum and its continuous spectrum both lie inside Γ. Then

$$\frac{1}{2\pi i}\int_\Gamma \chi(\lambda)(\lambda I-N)^{-k}d\lambda=0\quad (k=1,2,3,\cdots).$$

Proof. Let $\{K(\lambda)\}$ be the complex spectral family of N, and D the union of the point spectrum and the continuous spectrum of N. Since, as will be found immediately from the lemma quoted in the proof of Theorem 30,

$$\frac{1}{2\pi i}\int_\Gamma \chi(\lambda)(\lambda-z)^{-k}d\lambda=0\quad (k=1,2,3,\cdots)$$

for every point z inside Γ, and since, by hypotheses, every point $z\in D$ lies within Γ, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma}\chi(\lambda)(\lambda I-N)^{-k}d\lambda=\frac{1}{2\pi i}\int_{\Gamma}\chi(\lambda)\int_{D}(\lambda-z)^{-k}dK(z)d\lambda$$
$$=\int_{D}\left\{\frac{1}{2\pi i}\int_{\Gamma}\chi(\lambda)(\lambda-z)^{-k}d\lambda\right\}dK(z)$$
$$=0$$

for $k=1,2,3,\cdots$, as we wished to prove.

Theorem 32. Let $T(\lambda)$, $R(\lambda)$, Γ, and N be the same notations as before respectively. Then

$$\frac{1}{2\pi i}\int_{\Gamma}T(\lambda)(\lambda I-N)^{-k}d\lambda=R^{(k-1)}(N)/(k-1)! \quad (k=1,2,3,\cdots).$$

Proof. An application of Theorem 31 yields at once the result that

$$\frac{1}{2\pi i}\int_{\Gamma}T(\lambda)(\lambda I-N)^{-k}d\lambda=\frac{1}{2\pi i}\int_{\Gamma}R(\lambda)(\lambda I-N)^{-k}d\lambda$$
$$=\int_{D}\frac{R^{(k-1)}(z)}{(k-1)!}dK(z)$$
$$=\frac{R^{(k-1)}(N)}{(k-1)!} \quad (k=1,2,3,\cdots).$$

Theorem 33. Let $T(\lambda)$ be the same notation as in Theorem 30, $\Psi(\lambda)$ the second principal part of $T(\lambda)$, and ρ an arbitarily given positive number such that $\bigcup_{j=1}^{n}D_j$ and the closure of $\{\lambda_\nu\}$ are wholly contained in the domain $\{\lambda:|\lambda|<\rho\}$. Then

$$\Psi\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{1}{2\pi}\int_0^{2\pi}T(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt$$
$$-\frac{1}{2\pi}\int_0^{2\pi}T(\rho e^{it})\frac{e^{it}}{e^{it}-\kappa e^{i\theta}}dt-\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}c_{\alpha}^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_\nu\right)^{-\alpha}$$

for every κ with $0<\kappa<1$.

Proof. Let $\Phi(\lambda)$ denote the first principal part of $T(\lambda)$. Then $\Phi(\lambda)$ is expressed in the form of $\Phi(\lambda)=\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}c_{\alpha}^{(\nu)}/(\lambda-\lambda_\nu)^{\alpha}$ and it is found with the aid of (30) that

$$\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{T(\lambda)}{\lambda}d\lambda=R(0),$$

while

$$\frac{1}{2\pi i}\int_{|\lambda|=\rho}T(\lambda)(\lambda-z)^{-1}d\lambda=-\{\Phi(z)+\Psi(z)\}$$

for every point z ($\neq\infty$) outside the circle $|\lambda|=\rho$ oriented positively. Hence, by replacing $S(\lambda)$ in the proof of Theorem 4 by $T(\lambda)$ and then by following the argument used there [cf. Proc. Japan Acad., Vol. 38, No. 8, 452–454 (1962)], we can verify the validity of the present theorem.

Corollary 6. Let $T(\lambda)$ and ρ be the same notations as those in the preceding theorem respectively, and $R(\lambda)$ the ordinary part of $T(\lambda)$. If there exist countably infinite points $r_j e^{i\theta_j}$ $(j=1,2,3,\cdots)$ within the circle $|\lambda|=\rho$ such that

$$\int_0^{2\pi} \frac{T(\rho e^{it})}{\rho e^{it}-r_j e^{i\theta_j}}dt=0 \quad (j=1,2,3,\cdots),$$

then

$$T\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{1}{2\pi}\int_0^{2\pi} T(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt \quad (0<\kappa<1),$$

where the complex Poisson integral of $T(\lambda)$ on the right converges uniformly to $R(0)$ or to $T(\rho e^{i\theta})$ according as κ tends to zero or to unity.

Proof. By reasoning exactly like that used to prove Corollary 1 [cf. loc. cit., 454–455 (1962)], we can establish the present corollary.

Remark. In this case it is verified by the additional supposition on $T(\lambda)$ that $R(\lambda)$ is a constant.

Theorem 34. Let $T(\lambda)$ and ρ be the same notations as those in Theorem 33 respectively; Let $R(\lambda)$ be the ordinary part of $T(\lambda)$; and let

$$(31)\quad a_p=\frac{1}{\pi}\int_0^{2\pi} T(\rho e^{it})\cos pt\,dt,\quad b_p=\frac{1}{\pi}\int_0^{2\pi} T(\rho e^{it})\sin pt\,dt\ (p=0,1,2,\cdots).$$

Then

$$R(\kappa\rho e^{i\theta})=\frac{a_0}{2}+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)(\kappa e^{i\theta})^p \quad (\theta\text{: variable})$$

for every κ with $0\leqq\kappa<\infty$ and the series on the right converges absolutely and uniformly.

Proof. If we replace $S(\lambda)$ in the proof of Theorem 6 by $T(\lambda)$ [cf. Proc. Japan Acad., Vol. 38, No. 9, 641 (1962)], the reasoning used there can be applied without change to show the validity of the present theorem.

Theorem 35. Let $T(\lambda)$ and ρ be the same notations as in Theorem 33, and let a_p and b_p be given by (31). Then

$$(32)\quad T\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)\left(\frac{e^{i\theta}}{\kappa}\right)^p+\frac{1}{2}\sum_{p=1}^{\infty}(a_p+ib_p)\left(\frac{\kappa}{e^{i\theta}}\right)^p$$

for every κ with $0<\kappa<1$ and the two series on the right converge absolutely and uniformly.

Proof. This theorem also can be verified immediately from replacing $S(\lambda)$ in the proof of Theorem 7 by $T(\lambda)$ [cf. loc. cit., 641–642 (1962)].

Theorem 36. Let $T(\lambda)$ and ρ be the same notations as before. If the ordinary part of $T(\lambda)$ is a constant C, then

$$T\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\sum_{p=1}^{\infty}a_p\left(\frac{\kappa}{e^{i\theta}}\right)^p \quad \left(0<\kappa<1,\ \frac{a_0}{2}=C\right),$$

where a_p is given by (31) for every value of $p=0,1,2,\cdots$.

Proof. If we replace $S(\lambda)$ in the proof of Theorem 9 by $T(\lambda)$ [cf. loc. cit., 643–645 (1962)], we also can establish the present theorem by following the argument used there.

Remark. In this case the relation $a_p=ib_p$ holds for $p=1,2,3,\cdots$.

Theorem 37. Let $T(\lambda)$, Γ, ρ, a_p, and b_p be the same notations as before, and $R(\lambda)$ the ordinary part of $T(\lambda)$. If $R'(0)\neq 0$, then

$$\frac{1}{2\pi i}\int_\Gamma T(\lambda)d\lambda=\frac{a_1^2+b_1^2}{4R'(0)},$$

where $a_1^2+b_1^2$ remains constant for all finite values of ρ subject to the hypothesis that the union of the set $\bigcup_{j=1}^{n} D_j$ and the closure of $\{\lambda_\nu\}$ is wholly contained within the circle $|\lambda|=\rho$.

Proof. In the same manners as those used to prove Lemma A and Theorem 10 [cf. Proc. Japan Acad., Vol. 38, No. 9, 646–647 (1962)], we can show the present theorem.

Remark. If $\Phi(\lambda)$ denotes the first principal part of $T(\lambda)$, $\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)d\lambda=\sum_{\nu=1}^{\infty} c_1^{(\nu)}$.

Furthermore, if we consider such a ρ as was defined above for $T(\lambda)$ instead of considering a positive number σ with $\sup_\nu|\lambda_\nu|<\sigma<\infty$ for $S(\lambda)$, it can be verified that the results in all other theorems already established for $S(\lambda)$ are also valid for $T(\lambda)$; because the methods of the proofs of those theorems are based on the expansion of $S(\lambda)$ which has the same form as that of the expansion of $T(\lambda)$. In the case where all the accumulation points of $\{\lambda_\nu\}$ form a countable set, however, it is to be noted that the second principal part of $S(\lambda)$ vanishes, while that of $T(\lambda)$ never vanishes.