107. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XII

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In the preceding papers we have been concerned with the general method of constructing normal operators with arbitrarily prescribed point spectra in the complex abstract Hilbert space $\mathfrak{H}$ being complete, separable, and infinite dimensional and with its applications to the theory of functions of a complex variable. In the present paper, however, we shall first set ourselves the problem of constructing normal operators with arbitrarily prescribed continuous spectra in $\mathfrak{H}$.

Theorem 29. Let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrary bounded infinite sequence of complex numbers (counted according to the respective multiplicities), and D an arbitrary connected close set with positive finite measure in the complex plane such that any point of the closure of $\{\lambda_\nu\}$ is not contained in it. Then there are infinitely many bounded normal operators N in $\mathfrak{H}$ such that the point spectrum and the continuous spectrum of each N are given respectively by $\{\lambda_\nu\}$ and the union of D and the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself.

Proof. Let Δ be a Lebesgue-measurable set of positive finite measure $m(\Delta)$ in the complex λ-plane such that it contains D as its proper subset with measure less than $m(\Delta)$, and $L_2(\Delta)$ the Lebesgue functionspace associated with Δ. If we consider the operator T defined by $(Tf)(\lambda)=\lambda f(\lambda)$ for every $f\in L_2(\Delta)$, then it can be verified without difficulty that the adjoint operator T^* of T is given by $(T^*f)(\lambda)=\bar{\lambda}f(\lambda)$ and that T is a bounded normal operator in $L_2(\Delta)$ such that its continuous spectrum and its point spectrum are given by $\bar{\Delta}$ and the empty set respectively. Suppose now that $\{K(\lambda)\}$ denotes the complex spectral family of T and that $\{\hat{\varphi}_\nu(\lambda)\}$ and $\{\hat{\psi}_\mu(\lambda)\}$ are arbitrarily chosen orthonormal sets determining the subspaces $K(\bar{\Delta}-D)L_2(\Delta)=\hat{\mathfrak{M}}$ and $K(D)L_2(\Delta)=\hat{\mathfrak{N}}$ respectively. If we divide D in n disjoint subsets $D_1, D_2,\cdots, D_n$ with positive measure, then $K(D_i)K(D_j)=0$ $(i\neq j;\ i,j=1,2,3,\cdots,n)$ and any $K(D_j)$ is never the null operator because of the fact that D_j belongs to the continuous spectrum of T. Hence the dimension of the space $K(D)L_2(\Delta)$ is greater than n, however large n may be. The same is true of $K(\bar{\Delta}-D)L_2(\Delta)$. Since, moreover, $L_2(\Delta)=\hat{\mathfrak{M}}\oplus\hat{\mathfrak{N}}$, it turns out from these results that $\{\hat{\varphi}_\nu(\lambda)\}$ and

$\{\hat{\psi}_\mu(\lambda)\}$ both are infinite sets and together construct a complete orthonormal system in $L_2(\Delta)$. If, for brevity of expression, we now put

$$\hat{N}_1=\sum_{\nu=1}^{\infty}\lambda_\nu\hat{\varphi}_\nu(\lambda)\otimes L_{\hat{\varphi}_\nu(\lambda)}\quad [L_{\hat{\varphi}_\nu(\lambda)}(\,\cdot\,)=(\,\cdot\,,\hat{\varphi}_\nu(\lambda))],$$
$$\hat{N}_2=TK(D)$$

and consider the operator $\hat{N}$ defined by $\hat{N}=\hat{N}_1+\hat{N}_2$, then it is shown as below that $\hat{N}$ is a bounded normal operator in $L_2(\Delta)$ and that the point spectrum and the continuous spectrum of $\hat{N}$ are given respectively by $\{\lambda_\nu\}$ and the union of D and the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself. In the first place it is readily verified by direct computation that the adjoint operator $\hat{N}_1^*$ of $\hat{N}_1$ is given by

$$\hat{N}_1^*=\sum_{\nu=1}^{\infty}\bar{\lambda}_\nu\hat{\varphi}_\nu(\lambda)\otimes L_{\hat{\varphi}_\nu(\lambda)}$$

and hence that $\hat{N}_1$ is a bounded normal operator in $L_2(\Delta)$ such that its point spectrum consists of $\{0\}\cup\{\lambda_\nu\}$, $\hat{\mathfrak{N}}$ is the eigenspace of $\hat{N}_1$ corresponding to the eigenvalue $0\notin\{\lambda_\nu\}$, and its continuous spectrum consists of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself. In the next place it is easily verified by the orthogonality of $K(D)$ and $K(\bar{\Delta}-D)$ and the commutability of T^* and $K(D)$ that

$$(\hat{N}_2f)(\lambda)=\begin{cases}0 & \text{(for every } f\in\hat{\mathfrak{M}} \text{ and almost every } \lambda\in\Delta)\\ \lambda f(\lambda) & \text{(for every } f\in\hat{\mathfrak{N}} \text{ and almost every } \lambda\in\Delta)\end{cases}$$

and

$$(\hat{N}_2^*f)(\lambda)=\begin{cases}0 & \text{(for every } f\in\hat{\mathfrak{M}} \text{ and almost every } \lambda\in\Delta)\\ \bar{\lambda} f(\lambda) & \text{(for every } f\in\hat{\mathfrak{N}} \text{ and almost every } \lambda\in\Delta),\end{cases}$$

where $\hat{N}_2^*$ denotes the adjoint operator of $\hat{N}_2$. Since, in addition, $\hat{N}_2=\int_{\bar{\Delta}\cap D}\lambda dK(\lambda)$, it therefore is found that $\hat{N}_2$ is a bounded normal operator with continuous spectrum D in $L_2(\Delta)$ such that its point spectrum consists only of $\{0\}$ with corresponding eigenspace $\hat{\mathfrak{M}}$.

These results established above enable us to assert that $\hat{N}$ is a bounded normal operator in Hilbert space $L_2(\Delta)$ such that its point spectrum and its continuous spectrum are given respectively by $\{\lambda_\nu\}$ and the union of D and the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself, as we were to show. In addition, any $\hat{\varphi}_\nu$ is an eigenelement of $\hat{N}$ corresponding to the eigenvalue λ_ν. Accordingly the application of the functional-representation of a normal operator in Hilbert space leads us to the conclusion that

$$\widehat{N}=\sum_{\nu=1}^{\infty}\lambda_\nu\widehat{\varphi}_\nu(\lambda)\otimes L_{\widehat{\varphi}_\nu(\lambda)}+\sum_{\mu=1}^{\infty}\widehat{\Psi}_\mu(\lambda)\otimes L_{\widehat{\psi}_\mu(\lambda)},$$

where $\widehat{\Psi}_\mu(\lambda)=\sum_{j=1}^{\infty}(\widehat{N}\widehat{\psi}_\mu, \widehat{\psi}_j)\widehat{\psi}_j(\lambda)=\sum_{j=1}^{\infty}\int_\Delta \lambda\widehat{\psi}_\mu(\lambda)\overline{\widehat{\psi}_j(\lambda)}d\lambda\cdot\widehat{\psi}_j(\lambda)$ [cf. Proc. Japan Acad., Vol. 39, No. 10, 743–748 (1963)].

After these preliminaries, we shall turn to the proof of the theorem.

For brevity of expression, we put $(\widehat{N}\widehat{\psi}_i, \widehat{\psi}_j)=\beta_{ij}$ and denote by (β_{ij}) the matrix-operator in Hilbert coordinate space l_2 corresponding to an infinite matrix where β_{ij} is the element appearing in row i column j. Since $\widehat{N}$ is a bounded normal operator in Hilbert space $L_2(\Delta)$, (β_{ij}) here is a bounded normal operator with $\sum_{j=1}^{\infty}|\beta_{ij}|^2>|\beta_{ii}|^2>0$ $(i=1,2,3,\cdots)$ in l_2 by virtue of theorem C [cf. loc. cit., 746–748]. If we now consider the operator N defined by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu} \qquad \Big(\Psi_\mu=\sum_{j=1}^{\infty}\beta_{\mu j}\psi_j\Big),$$

where $\{\varphi_\nu\}$ and $\{\psi_\mu\}$ are arbitrarily given incomplete infinite orthonormal sets orthogonal to each other such that $\mathfrak{H}$ is determined by them, then N is therefore a bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $\mathfrak{H}$ according to Theorem B [cf. loc. cit., 744–746 and (B) in Remark of the present paper]. Thus it will suffice to prove that the continuous spectrum of N coincides with that of $\widehat{N}$.

If, for the sake of simplicity, we put $\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}=N_2$, then $N_2\psi_\kappa=\Psi_\kappa=\sum_{j=1}^{\infty}\beta_{\kappa j}\psi_j$, and hence

$$\begin{aligned}(N_2\psi_\kappa, \psi_j)&=\beta_{\kappa j}\\&=(\widehat{N}_2\widehat{\psi}_\kappa, \widehat{\psi}_j)\end{aligned}$$

for every pair of κ, $j=1,2,3,\cdots$. Furthermore, by applying the final relations and the fact that any $\widehat{N}_2\widehat{\psi}_\kappa$, $(\kappa=1,2,3,\cdots)$, is orthogonal to every $\widehat{\varphi}_\nu\in\{\widehat{\varphi}_\nu\}$, we have

$$\begin{aligned}((\lambda I-\widehat{N}_2)\widehat{\psi}_\kappa(z),\ (\lambda I-\widehat{N}_2)\widehat{\psi}_j(z))&=|\lambda|^2(\widehat{\psi}_\kappa, \widehat{\psi}_j)-\bar{\lambda}(\widehat{N}_2\widehat{\psi}_\kappa, \widehat{\psi}_j)-\lambda\overline{(\widehat{N}_2\widehat{\psi}_j, \widehat{\psi}_\kappa)}\\&\quad+\sum_{\mu=1}^{\infty}(\widehat{N}_2\widehat{\psi}_\kappa, \widehat{\psi}_\mu)(\widehat{\psi}_\mu, \widehat{N}_2\widehat{\psi}_j)\\&=|\lambda|^2(\psi_\kappa, \psi_j)-\bar{\lambda}(N_2\psi_\kappa, \psi_j)-\lambda\overline{(N_2\psi_j, \psi_\kappa)}\\&\quad+\sum_{\mu=1}^{\infty}(N_2\psi_\kappa, \psi_\mu)(\psi_\mu, N_2\psi_j)\\&=((\lambda I-N_2)\psi_\kappa,\ (\lambda I-N_2)\psi_j)\end{aligned}$$

and $(\widehat{\varphi}_\kappa(z), (\lambda I-\widehat{N}_2)\widehat{\psi}_j(z))=(\varphi_\kappa, (\lambda I-N_2)\psi_j)=0.$

If we here consider the element $f=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b_\mu\psi_\mu$ in $\mathfrak{H}$ corresponding to an arbitrary element $\widehat{f}=\sum_{\nu=1}^{\infty}a_\nu\widehat{\varphi}_\nu+\sum_{\mu=1}^{\infty}b_\mu\widehat{\psi}_\mu$ in $L_2(\Delta)$, then the just established relations yield the result that

$$\begin{aligned}||(\lambda I-\widehat{N})\widehat{f}(z)||^2&=\sum_{\nu=1}^{\infty}|a_\nu|^2|\lambda-\lambda_\nu|^2+||(\lambda I-\widehat{N}_2)\sum_{\mu=1}^{\infty}b_\mu\widehat{\psi}_\mu||^2\\&=\sum_{\nu=1}^{\infty}|a_\nu|^2|\lambda-\lambda_\nu|^2+||(\lambda I-N_2)\sum_{\mu=1}^{\infty}b_\mu\psi_\mu||^2\\&=||(\lambda I-N)f||^2.\end{aligned}$$

On the other hand, as is easily seen, a necessary and sufficient condition that λ be a point belonging to the resolvent set of N is that there exist a positive constant c with $||\lambda I-N||>c$. Consequently the relation $||\lambda I-\widehat{N}||=||\lambda I-N||$ derived from the last result above implies that the resolvent set of N coincides with that of $\widehat{N}$. In addition to it, N and $\widehat{N}$ have the same point spectrum $\{\lambda_\nu\}$, as was shown before. Thus the continuous spectrum of N coincides with that of $\widehat{N}$, so that the continuous spectrum of N consists of the union of D and the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself.

From the hypotheses that the incomplete infinite orthonormal sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$ are arbitrary, it is at once obvious that there exist infinitely many bounded normal operators N satisfying the conditions given in the statement of the present theorem.

The theorem has thus been proved.

Remark. The special case where the measure of Δ is linear is admissible. In that case any inner product for $L_2(\Delta)$ is a line integral along Δ. From a point of view of the function theory, we are rather interested in the case where D is a simple close curve. Supposing that both $f\in\mathfrak{H}$ and $f'\in\mathfrak{H}$ are arbitrary elements not orthogonal to every $\varphi_\nu\in\{\varphi_\nu\}$ and $\psi_\mu\in\{\psi_\mu\}$, in this case the function $F(\lambda)=((\lambda I-N)^{-1}f, f')$ of a complex variable λ is regular in the entire complex λ-plane except for the union of D and the closure of $\{\lambda_\nu\}$ and has D as its natural boundary in the sense of the classical theory of functions. If, in particular, the closure of $\{\lambda_\nu\}$ is a closed curve, then it also forms the natural boundary of $F(\lambda)$. Accordingly there are two kinds of natural boundaries.

(A) Remark on "A note on the functional-representations of normal operators in Hilbert spaces" (Proc. Japan Acad., Vol. 39, No. 9 (1963)) and "Some applications of the functional-representations of normal operators in Hilbert spaces. X" (Proc. Japan Acad., Vol. 40, No. 5 (1964)) by S. Inoue:

In the latter paper I stated that "the point spectrum of $\widetilde{N}[I-K(\Delta^{-}(\widetilde{N}))]$ is given by $\{\lambda_\nu\}$". However this expression was very ambiguous, since the domain considered for the operator was not assigned. Though, as far as the domain considered for the operator is restricted to the space $[I-K(\Delta^{-}(\widetilde{N}))]\mathfrak{H}$, the point spectrum consists of $\{\lambda_\nu\}$ alone for the restricted domain, it consists of $\{\lambda_\nu\}\cup\{0\}$ for the whole space $\mathfrak{H}$ provided that $K(\Delta^-(\tilde{N}))$ is not the null operator, and here $K(\Delta^-(\tilde{N}))\mathfrak{H}$ is the eigenspace of $\tilde{N}[I-K(\Delta^-(\tilde{N}))]$ corresponding to the eigenvalue $0\notin\{\lambda_\nu\}$. Hence the continuous spectrum of $\tilde{N}[I-K(\Delta^-(\tilde{N}))]$ is given by $\Delta^+(\tilde{N})$ or by $\Delta^+(\tilde{N})-\{0\}$. These also are same of Remark B of the former paper above. At any rate, however, these have no effect on the theory itself treated there.

(B) Remark on "A note on the functional-representations of normal operators in Hilbert spaces. II" (Proc. Japan Acad., Vol. 39, No. 10 (1963)) by S. Inoue:

Every $\beta_{\mu\mu}$ in Theorems B and C is subject to the condition $\beta_{\mu\mu}\neq 0$, as can be found immediately from the methods of the proofs of those theorems. Moreover the additional hypothesis $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$ to (β_{ij}) in Theorem B can be rejected. In fact, for every pair of elements $f=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b_\mu\psi_\mu$ and $f'=\sum_{\nu=1}^{\infty}a'_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b'_\mu\psi_\mu$ in $\mathfrak{H}$, we find that

$$(NN^*f,f')=(N^*f,N^*f')$$
$$=\sum_{\nu=1}^{\infty}|\lambda_\nu|^2a_\nu\bar{a}'_\nu+(N_2^*\sum_{\mu=1}^{\infty}b_\mu\psi_\mu,\ N_2^*\sum_{\mu=1}^{\infty}b'_\mu\psi_\mu)\quad (N_2=c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\phi_\mu}),$$

where $(N_2^*\psi_\mu, N_2^*\psi_\kappa)=\sum_{j=1}^{\infty}(N_2^*\psi_\mu,\psi_j)(\psi_j,N_2^*\psi_\kappa)=|c|^2\sum_{j=1}^{\infty}\bar{\beta}_{j\mu}\beta_{j\kappa}$, and similarly that

$$(N^*Nf,f')=(Nf,Nf')$$
$$=\sum_{\nu=1}^{\infty}|\lambda_\nu|^2a_\nu\bar{a}'_\nu+(N_2\sum_{\mu=1}^{\infty}b_\mu\psi_\mu,\ N_2\sum_{\mu=1}^{\infty}b'_\mu\psi_\mu),$$

where $(N_2\psi_\mu,N_2\psi_\kappa)=|c|^2\sum_{j=1}^{\infty}\beta_{\mu j}\bar{\beta}_{\kappa j}$. On the other hand, the equality

$$\sum_{j=1}^{\infty}\bar{\beta}_{j\mu}\beta_{j\kappa}=\sum_{j=1}^{\infty}\beta_{\mu j}\bar{\beta}_{\kappa j}\neq\infty$$

holds in accordance with the hypothesis that (β_{ij}) is a bounded normal matrix. Consequently we obtain the relation $(NN^*f,f')=(N^*Nf,f')$ which implies that $NN^*=N^*N$ in $\mathfrak{H}$. Thus the additional hypothesis $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$ is not needful to show the normality of N.