106. On the Summability Method (Y)

By Kazuo ISHIGURO
Department of Mathematics, Hokkaido University, Sapporo
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§1. When a sequence $\{s_n\}$ is given we consider the transformation

$$y_n=\frac{1}{2}(s_{n-1}+s_n) \quad (n=0,1,\cdots), \tag{1}$$

where $s_{-1}=0$. If the sequence $\{y_n\}$ tends to a finite limit s, $\{s_n\}$ is said to be summable (Y) to s. This method of summability was studied by O. Szász [4] in detail. G. H. Hardy also remarked this method in his book [1]. As is easily seen, this method is very similar to the ordinary convergence. However, it possesses some interesting properties.

By modifying this method slightly, we obtain the method of summability (Y^*) with the transformation

$$y_n^*=\frac{1}{2}(s_n+s_{n+1}) \quad (n=0,1,\cdots).$$

Obviously the methods (Y) and (Y^*) are equivalent. O. Szász [5] proved that the Borel summability (B) does not imply the product summability $(B\cdot Y^*)$.

Recently, W. K. Hayman and A. Wilansky [2] used the method (Y) to construct some counter example. In this note, we shall study these methods furthermore.

§2. We shall prove the following

Theorem 1. *If $\{s_n\}$ is Abel summable (A) to s, then it is also summable $(A\cdot Y)$ to the same sum. Here Y may be replaced by Y^*.*

Proof. The assertion follows from the equality

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty}y_nx^n&=(1-x)\cdot\frac{1}{2}\cdot\sum_{n=0}^{\infty}(s_{n-1}+s_n)x^n\\&=\frac{(1-x)}{2}\left\{\sum_{n=0}^{\infty}s_{n-1}x^n+\sum_{n=0}^{\infty}s_nx^n\right\}\\&=\frac{(1-x)(1+x)}{2}\sum_{n=0}^{\infty}s_nx^n.\end{aligned}$$

In the case of Y^*, the proof is quite similar.

It is interesting to remark that (B) implies[1] $(B\cdot Y)$ but (B) does not imply $(B\cdot Y^*)$ (see O. Szász [5]).

As a converse of the above theorem, we shall prove the following

1) Given two summability methods (P), (Q), we say that (P) *implies* (Q) if any sequence which is summable (P) is summable (Q) to the same sum.

Theorem 2. *If* $\{s_n\}$ *satisfies*

$$\lim_{n\to\infty} s_n x^n = 0$$

for all x *with* $0<x<1$, *and if* $\{s_n\}$ *is summable* $(A\cdot Y)$ *to* s, *then it is summable* (A) *to the same value. Here* Y *may be replaced by* Y^*.

Proof. From the assumption of the theorem, we have, for $0<x<1$,

$$\begin{aligned}\sum_{n=0}^{\infty}(s_{n-1}+s_n)x^n &= \lim_{m\to\infty}\sum_{n=0}^{m}(s_{n-1}+s_n)x^n \\ &= \lim_{m\to\infty}\left\{\sum_{n=0}^{m-1}s_n(1+x)x^n+s_m x^m\right\} \\ &= (1+x)\sum_{n=0}^{\infty}s_n x^n.\end{aligned}$$

Hence the assertion follows easily. In the case of Y^*, the proof is quite similar.

§3. As a weak summability method we know the harmonic means (h). This method is defined by means of the tranformation

$$h_n = \sum_{\nu=0}^{n}\frac{s_{n-\nu}}{\nu+1}\Bigg/\sum_{\nu=0}^{n}\frac{1}{\nu+1}.$$

See, e.g., G. H. Hardy [1] p. 110. Here we shall study the relation between the methods (Y) and (h). As is well known, these two methods are special cases of the Nörlund methods of summability.

We suppose that

$$p_n\geq 0,\ p_0>0,\ P_n=p_0+p_1+\cdots+p_n,$$

and define t_n by

$$t_n=\sum_{\nu=0}^{n}p_\nu s_{n-\nu}/P_n.$$

If $\{t_n\}$ tends to a finite limit s, $\{s_n\}$ is said to be summable (N, p_n) to s.

If $p_0=p_1=1$, and the remaining p_n are 0, then we obtain the method (Y). If $p_n=1/(n+1)$ $(n=0,1,\cdots)$, then we obtain the method (h). See, e.g., G. H. Hardy [1] p. 64.

Here we shall prove the following theorems.

Theorem 3. (h) *does not imply* (Y).

Proof. Now, suppose $s_{3m}=1$ $(m=0,1,\cdots)$, and the remaining s_n are 0. Then we have

$$h_0=1,\ h_1=\frac{\frac{1}{2}}{1+\frac{1}{2}},\ h_2=\frac{\frac{1}{3}}{1+\frac{1}{2}+\frac{1}{3}},$$

$$h_3=\frac{1+\frac{1}{4}}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}},\ h_4=\frac{\frac{1}{2}+\frac{1}{5}}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}},\cdots.$$

We see easily

$$\lim_{m\to\infty} h_{3m} = \lim_{m\to\infty} h_{3m+1} = \lim_{m\to\infty} h_{3m+2} = \frac{1}{3},$$

hence

$$\lim_{n\to\infty} h_n = \frac{1}{3}.$$

Next we have

$$y_0 = \frac{1}{2},\ y_1 = \frac{1}{2},\ y_2 = 0, \cdots.$$

Thus

$$\lim_{m\to\infty} y_{3m} = \lim_{m\to\infty} y_{3m+1} = \frac{1}{2},\ \lim_{m\to\infty} y_{3m+2} = 0,$$

and $\{y_n\}$ does not converge.

Theorem 4. (Y) *does not imply* (h).

Proof. We shall use the fundamental theorem on the Nörlund methods (see G. H. Hardy [1], Theorem 19). Following Hardy's notations, let

$$p(x) = 1 + x,$$

$$q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} \quad (|x| < 1),$$

$P_0 = 1,\ P_2 = P_3 = \cdots = 2,$

$$Q_n = \sum_{\nu=0}^{n} \frac{1}{\nu+1},$$

$$k(x) = \sum_{n=0}^{\infty} k_n x^n = \frac{q(x)}{p(x)}$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n+1}\right),$$

and

$$k_n = \frac{1}{n+1} - \frac{1}{n} + \cdots + (-1)^n.$$

Since

$$|k_n| = (-1)^n k_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1},$$

$\{|k_n|\}$ tends to $\log 2$. Now,

$$|k_0| P_n + |k_1| P_{n-1} + \cdots + |k_n| P_0 >$$
$$> |k_0| + |k_1| + \cdots + |k_n| = K_n, \text{ say.}$$

We shall prove here that

$$K_n \leq H Q_n$$

does not hold for sufficiently large n, how large H may be taken.

Otherwise, since

$$|k_n| > \frac{1}{2} \log 2$$

for $n > N$, say, and

$$K_n > \frac{1}{2}(n - N) \log 2,$$

we would obtain

$$\frac{1}{2}(n-N)\log 2<HQ_n$$

and

$$n<H\log n^{2)} \quad \text{for} \quad n>N.$$

But this is impossible, whence the proof is complete according to the fundamental theorem.

§4. We shall prove here the following

Theorem 5. (Y) *implies* (B).

Proof. From (1) we get

$$s_n=2\{y_n-y_{n-1}+\cdots+(-1)^n y_0\},$$

and

$$\begin{aligned} B(x)&=e^{-x}\sum_{n=0}^{\infty}s_n\frac{x^n}{n!}\\ &=2e^{-x}\sum_{n=0}^{\infty}\{y_n-y_{n-1}+\cdots+(-1)^n y_0\}\frac{x^n}{n!}\\ &=2e^{-x}\left\{y_0\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}-y_1\sum_{n=1}^{\infty}\frac{(-x)^n}{n!}+\cdots+\right.\\ &\qquad\left.+(-1)^{\nu}y_{\nu}\sum_{n=\nu}^{\infty}\frac{(-x)^n}{n!}+\cdots\right\}, \end{aligned}\tag{2}$$

where the last equality may be justified from the boundedness of $\{y_n\}$. See, e.g., E. W. Hobson [3], p. 52.

Here we put

$$\varphi_{\nu}(x)=2e^{-x}(-1)^{\nu}\sum_{n=\nu}^{\infty}\frac{(-x)^n}{n!},$$

then

$$B(x)=y_0\varphi_0(x)+y_1\varphi_1(x)+\cdots+y_{\nu}\varphi_{\nu}(x)+\cdots.$$

We shall prove that if $\{y_n\}$ tends to s, then $B(x)$ tends to the same value for $x\to\infty$.

(i) We see immediately

$$\lim_{x\to\infty}\varphi_{\nu}(x)=0 \quad (\nu=0,1,\cdots).$$

(ii) For the same reason as in (2), we obtain

$$\begin{aligned} \sum_{\nu=0}^{\infty}\varphi_{\nu}(x)&=2e^{-x}\sum_{\nu=0}^{\infty}(-1)^{\nu}\sum_{n=\nu}^{\infty}\frac{(-x)^n}{n!}\\ &=2e^{-x}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\sum_{\nu=0}^{n}(-1)^{\nu}\\ &=2e^{-x}\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\\ &=2e^{-x}\cdot\frac{e^x+e^{-x}}{2}\\ &=1+e^{-2x}. \end{aligned}$$

2) We use H to denote a constant, possibly different at each occurrence.

Hence

$$\lim_{x\to\infty}\sum_{\nu=0}^{\infty}\varphi_\nu(x)=1.$$

(iii) We see easily

$$\varphi_\nu(x)>0 \quad \text{for} \quad x>0 \qquad (\nu=0,1,\cdots).$$

Hence, for $x>0$,

$$\sum_{\nu=0}^{\infty}|\varphi_\nu(x)|=\sum_{\nu=0}^{\infty}\varphi_\nu(x)=1+e^{-2x}<2.$$

Collecting the above estimations, we obtain the desired conclusion from the fundamental theorem on sequence-to-function transformations. See, e.g., G. H. Hardy [1], Theorem 5.

Further, we obtain the following

Theorem 6. *There is a sequence summable* (B) *but not summable* (Y).

Proof. To prove (B) does not imply $(B\cdot Y^*)$, O. Szász [5] used the sequence $\{s_n\}$ defined by

$$e^{-x}\sum_{n=0}^{\infty}s_n\frac{x^n}{n!}=\int_0^x \cos(e^t)\,dt.$$

This sequence has the desired property. Otherwise, it would be summable (Y^*) also. From the regularity of the method (B), it would be also summable $(B\cdot Y^*)$, but it is impossible.

Remark. It is interesting to note that (Y^*) implies (B) but (B) does not imply $(B\cdot Y^*)$.

References

[1] G. H. Hardy: Divergent Series. Oxford (1949).
[2] W. K. Hayman and A. Wilansky: An example in summability. Bull. Amer. Math. Soc., **67**, 554–555 (1961).
[3] E. W. Hobson: The Theory of Functions of a Real Variable and the Theory of Fourier's Series. vol. II, Cambridge (1927).
[4] O. Szász: Introduction to the Theory of Divergent Series. Cincinnati (1952).
[5] ——: On the product of two summability methods. Ann. Soc. Polon. Math., **25**, 75–84 (1952).