102. On the Normality of Certain Product Spaces

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Let X be the image of a metric space R under a closed continuous mapping f and let Y be the image of a metric space S under a closed continuous mapping g. We shall be concerned with the normality of the product space $X\times Y$.

As is well known, the spaces X and Y are both paracompact and perfectly normal. But the topological product of two normal spaces is not normal in general. In fact, as the example given by E. Michael [2] shows, the product space $W\times Z$ is not necessarily normal, even if W is a hereditarily paracompact Hausdorff space with Lindelöf property and Z is a separable metric space.

K. Morita has given in [4] two closed continuous mappings whose product is not a closed mapping. It should be noted that one of these mappings is a perfect mapping, and hence the product of a closed continuous mapping and a perfect mapping is not always a closed mapping. Thus the normality of the product space $X\times Y$ is not concluded directly from the normality of the product space $R\times S$.

In this note, we shall establish the following:

Theorem 1. *If the space R is a locally compact metric space, then the product space $X\times Y$ is normal.*

1. Our proof will be based on the following theorems established by K. Morita in [5] and [3].

Theorem 2. *Let X be a paracompact normal space which is a countable union of locally compact closed subsets, and let Y be a paracompact normal space. Then the product space $X\times Y$ is paracompact and normal.*

Theorem 3. *Let X be a paracompact and perfectly normal space, which is a countable union of locally compact closed subsets and is also a countable union of closed metrizable subspaces. Let Y be a paracompact and perfectly normal space. Then the product space $X\times Y$ is paracompact and perfectly normal.*

Theorem 4. *Let f be a closed continuous mapping of a paracompact and locally compact Hausdorff space R onto another topological space X. Denote by X' be the set of all points x of X such that $f^{-1}(x)$ is not compact, and by X'' the set of all points x of X such that $\mathfrak{B}f^{-1}(x)$ is not compact. Then we have:*

(a) $X''\subset X'$;

(b) *X' is a closed discrete subset of X;*

(c) *$X-X''$ is locally compact;*

(d) *the closure of any neighbourhood of x is not compact for every point x of X''.*

Here $\mathfrak{B}f^{-1}(x)$ denotes the boundary of the inverse image $f^{-1}(x)$ of the point x.

Remark. Recently, Edwin Duda has announced in Bulletin of the American Mathematical Society the results concerning to the upper semi-continuous decomposition of a locally compact separable metric space into closed sets [1]. But these results are already included in the K. Morita's Theorem 4 mentioned above. (The separability seems to be superfluous.)

2. Theorem 1 is generalized to the following

Theorem 5. *Let X be the image under a closed continuous mapping f of the paracompact and perfectly normal space R, and the space R be a countable union of locally compact and metrizable closed subspaces. Then, the product space $X\times Z$ is paracompact and perfectly normal for any paracompact and perfectly normal space Z.*

First of all, we shall prove the following

Theorem 6. *Let X be the image under a closed continuous mapping f of the paracompact and perfectly normal Hausdorff space R, and the space R be a countable union of locally compact closed subspaces. Then the product space $X\times Z$ is paracompact and normal for any paracompact and normal space Z.*

Proof. Let R be the union of the locally compact closed subspaces $R_n, n=1,2,\cdots$. Since R is a paracompact and perfectly normal Hausdorff space, and since each R_n is closed in R, each R_n is a locally compact, paracompact and perfectly normal Hausdorff subspace of R. Furthermore, the partial mapping $f|R_n$: $R_n\rightarrow f(R_n)=X_n(\subset X)$ is a closed continuous onto mapping. Hence, if we denote by X_n' the set of all points x of X_n such that the inverse image $f^{-1}(x)$ is not compact and by X_n'' the set of all points x of X_n such that the boundary of the inverse image $\mathfrak{B}f^{-1}(x)$ is not compact, we have, by Theorem 4,

(a) $X_n''\subset X_n'$;

(b) X_n' is a closed discrete subset of X_n;

(c) X_n-X_n'' is a locally compact subspace of X_n.

Therefore, X_n-X_n' is an open subset of a locally compact space X_n-X_n'', and hence X_n-X_n' is a locally compact open subspace of X_n. On the other hand, R_n is perfectly normal, and hence X_n is perfectly normal. Thus X_n-X_n' is an F_σ set in X_n, and we can put

$$X_n-X_n'=\bigcup_{i=1}^{\infty}F_{ni},$$

where each $F_{ni}, i=1,2,\cdots$, is a closed subset of X_n.

As $X_n - X'_n$ is locally compact, each F_{ni} is locally compact. Since the mapping f is closed continuous and each R_n is closed in R, each $X_n = f(R_n)$ is closed in X, and hence each F_{ni} is a locally compact closed subset of the space X. Obviously X'_n is a locally compact closed subspace of the space X_n, and hence it is a locally compact closed subspace of the space X. Therefore each X_n is a countable union of locally compact closed subsets of the space X.

As $X = f(R) = \bigcup_{n=1}^{\infty} f(R_n) = \bigcup_{n=1}^{\infty} X_n$ holds, the space X is a countable union of locally compact closed subsets of X. Since R is paracompact and perfectly normal and f is a closed continuous mapping, the space X is also paracompact and perfectly normal. Thus the product space $X \times Z$ is paracompact and normal by Theorem 2.

Proof of Theorem 5. The partial mapping $f|f^{-1}(F_{ni})$ is a closed continuos mapping onto F_{ni} and, since $F_{ni} \subset X_n - X'_n$, $f|f^{-1}(F_{ni})$ is a perfect mapping. Hence each F_{ni} is metrizable, as each R_n is metrizable in this case. Of course each X'_n is a metrizable closed subspace. Then the space X is also a countable union of closed metrizable subspaces, and Theorem 5 follows from Theorem 3.

Proof of Theorem 1. The space Y is paracompact and perfectly normal. Therefore the product space $X \times Y$ is perfectly normal and paracompact by Theorem 5. Theorem 1 then follows.

References

[1] Edwin Duda: A locally compact separable metric space is almost invariant under a closed mapping. Bulletin Amer. Math. Soc., **70**, 285–286 (1964).

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