101. On Boundary Value Problem for Parabolic Equations

By Reiko ARIMA

(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1964)

1. Introduction. Let us consider the parabolic equation

$$(1)\qquad \frac{\partial}{\partial t}u=Au \quad \text{in} \quad (0,T)\times\Omega$$

$$\left(A=\sum_{|\nu|\leqq 2b}a_\nu(t,x)\left(\frac{\partial}{\partial x}\right)^\nu,\ L=\frac{\partial}{\partial t}-A\right)$$

with the zero initial data and the general boundary data

$$(2)\qquad B_ju=f_j\ (j=1,\cdots,b) \quad \text{on} \quad (0,T)\times S$$

$$\left(\beta_j=\sum_{|\nu|\leqq r_j}b_{j\nu}(t,x)\left(\frac{\partial}{\partial x}\right)^\nu,\ 0\leqq r_j\leqq 2b-1\right),$$

where Ω is a domain in R^n surrounded by a hypersurface S.

Recently, this problem was treated by Eidelman for systems ([1]). Here we use his construction and estimates of kernels in the case of constant coefficients and Ω is a half space. We shall introduce an operator defined on the boundary which plays an analogous role to the Riemann-Liouville-operator which was used by Mihailov in one dimensional case ([2]), therefore we need not assume that all r_j coincide, which was assumed by Eidelman in case of non-convex region. Finally we have the estimates for the Green function.*)

I thank Prof. Mizohata very much for his kind advices and encouragements throughout this subject.

Now, let $\{\overline{V}\}_I$ be a finite covering of S and a point $x=(x_1,\cdots,x_n)$ of $\overline{V}$ be represented by a local coordinate $\overline{x}'=(\overline{x}_1,\cdots,\overline{x}_{n-1})$, such that $x_j=F_j(\overline{x}')$ $(j=1,\cdots,n)$, where $F_j(\overline{x}')$ is of class-C^s $(s=2b+1+\gamma,\gamma>0)$, and $\overline{\overline{x}}'=\overline{\overline{x}}'(\overline{x}')$ is class-C^s where $x\in\overline{V}\cap\overline{\overline{V}}$. Then we have a n-dimensional neighbourhood $\overline{U}\supset\overline{V}$, such that the transformation defined by $x_j=F_j(\overline{x}')+N_j(\overline{x}')\dot{x}$ $(j=1,2,\cdots,n)$ is one-to-one and of class-c^{s-1} between $x\in\overline{U}$ and $\overline{x}$, where $N_x=(N_1,\cdots,N_n)$ is the unit inner normal vecter at $x\in S$. Here we put $\widetilde{S}=\bigcup_I\overline{U}$.

Put $A_0(\eta+zN_x;t,x)=(-1)^b\sum_{|\nu|=2b}a_\nu(t,x)(\eta+zN_x)^\nu$ and $B_{0j}(\eta+zN_x;t,x)$ $=(i)^{r_j}\sum_{|\nu|=r_j}b_{j\nu}(t,x)(\eta+zN_x)^\nu$, where $\eta\in T_x=R^n/\{zN_x\}$, $z\in R^1$, $t\in(0,T)$, $x\in S$. Let $A_{0+}(p,\eta,z;t,x)$ be the polynomial of z of degree b (the coefficient of z^b is 1), where the roots are composed of all the roots z of $p-A_0$ $(\eta+zN_x;t,x)=0$, having the positive imaginary part. Then let us denote

*) Detailed proof will be published in a forthcoming paper.

$$R(p,\eta;t,x)=\det.\begin{pmatrix}\oint\frac{B_{01}(\eta+zN_x;t,x)}{A_{0+}(p,\eta,z;t,x)}dz\cdots\oint\frac{B_{01}(\eta+zN_x;t,x)z^{b-1}}{A_{0+}(p,\eta,z;t,x)}dz\\ \cdots\cdots\cdots\\ \oint\frac{B_{0b}(\eta+zN_x;t,x)}{A_{0+}(p,\eta,z;t,x)}dz\cdots\oint\frac{B_{0b}(\eta+zN_x;t,x)z^{b-1}}{A_{0+}(p,\eta,z;t,x)}dz\end{pmatrix}$$

where the integrals are taken along a closed curve in the complex-z plane surrounding all the roots of $A_{0+}=0$.

Now we assume the following assumptions

i) $\operatorname{Re} A_0(\sigma;t,x)<0$ $(\sigma\in R^n,\ \sigma\neq 0,\ t\in(0,T),\ x\in\Omega)$,

ii) $R(p,\eta;t,x)\neq 0$ $(\operatorname{Re} p\geqq 0,\ \eta\in T_x,\ (\rho,\eta)\neq 0,\ t\in(0,T),\ x\in S)$.

Concerning the regularity of the coefficients of L and B_j and that of f_j, we assume

iii) $a_\nu(t,x)\in C^{\gamma}(t,x)$ $(x\in\Omega)$ $(\gamma>0)$,
$b_{j\nu}(t,x)\in C^{2b\beta_j+\gamma}(t,x)$ $(x\in S)$,

iv) $f_j(t,x)\in C_0^{2b\beta_j+\gamma}(t,x)$ $(x\in S)$,

where $\beta_j=\alpha(2b-1-\varepsilon-r_j)$ $(\alpha=1/2b,\ 0<\varepsilon<\gamma$ in case of max. $r_j<2b-1$, $\varepsilon=0$ in case of max. $r_j=2b-1)$.

2. Functional spaces C^β, C_0^β, $\hat{C}_l^\beta$.

1. $C^\beta(t,x)$. $f(t,x)\in C^\beta(t,x)$ means

i) $$\left|\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}f(t,x)\right|\leqq C,\quad (2bk_0+|k|\leqq[\beta]),$$

ii) $$\left|\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}[f(t,x+\Delta)-f(t,x)]\right|\leqq C|\Delta|^{\beta-[\beta]}\quad (2bk_0+|k|=[\beta]),$$

iii) $$\left|\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}[f(t+\Delta,x)-f(t,x)]\right|\leqq C|\Delta|^{\alpha(\beta-2bk_0-|k|)},$$
$$([\beta]-2b<2bk_0+|k|\leqq[\beta],\ \alpha=1/2b).$$

2. $C_0^\beta(t,x)$. $f(t,x)\in C_0^\beta(t,x)$ means

i) $$f(t,x)\in C^\beta(t,x),$$

ii) $$f(t,x)=0\quad (t<0).$$

3. $\hat{C}_l^\beta(t,x)$. $f(t,x;\tau,\xi)\in\hat{C}_l^\beta(t-\tau,x-\xi)$ means

i) $$\left|\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}f(t,x;\tau,\xi)\right|\leqq C(t-\tau)^{-\alpha(l+2bk_0+|k|)}e^{-\psi(t-\tau,\,x-\xi)}$$

$(2bk_0+|k|\leqq[\beta])$, where $\psi(t,x)=c|x/t^\alpha|^q$, $\alpha=1/2b$ and $q=2b/2b-1$ (c is a positive constant).

ii) $$\left|\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}[f(t,x+\Delta;\tau,\xi)-f(t,x;\tau,\xi)]\right|$$
$$\leqq C|\Delta|^{\beta-[\beta]}(t-\tau)^{-\alpha(l+\beta)}e^{-\psi(t-\tau,\,x-\xi)}$$
$$(2bk_0+|k|=[\beta],\ |\Delta|<(t-\tau)^\alpha),$$

iii) $$\left|\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}[f(t+\Delta,x;\tau,\xi)-f(t,x;\tau,\xi)]\right|$$
$$\leqq C\Delta^{\alpha(\beta-2bk_0-|k|)}(t-\tau)^{-\alpha(l+\beta)}e^{-\psi(t-\tau,\,x-\xi)}$$
$$([\beta]-2b<2bk_0+|k|\leqq[\beta],\ 0<\Delta<t-\tau),$$

iv) $f(t, x; \tau, \xi)=0\ (t<\tau)$.

LEMMA 1. Assume $G(t, x; \tau, \xi)\in \widehat{C}^{\beta}_{n+2b-l}(t-\tau, x-\xi)\ (x, \xi\in R^n, 0\leqq\beta<l)$.

i) Let $f(t, x)\in C^0(t, x)$, then

$$\int_0^t d\tau\int G(t, x; \tau, \xi)f(\tau, \xi)\,d\xi\in C_0^{\beta}(t, x),$$

ii) Let $f(t, x; \tau_0, \xi_0)\in \widehat{C}^0_{l_1}(t-\tau_0, x-\xi_0)$, $(x, \xi_0\in R^n, l_1<n+2b)$, then

$$\int_{\tau_0}^t d\tau\int G(t, x; \tau, \xi)f(\tau, \xi; \tau_0, \xi_0)\,d\xi\in \widehat{C}^{\beta}_{l_1-l}(t-\tau_0, x-\xi_0).$$

3. **Operator K_β.** Let us consider the parabolic operator $\mathcal{L}$ defined on $(0, T)\times S$:

$$\mathcal{L}=\frac{\partial}{\partial t}+\left\{-\frac{1}{\sqrt{\overline{g}(\overline{x}')}}\sum_{i,j=1}^{n-1}\frac{\partial}{\partial \overline{x}_i}\sqrt{\overline{g}(\overline{x}')}\,\overline{g}^{ij}(\overline{x}')\frac{\partial}{\partial \overline{x}_j}\right\}^b$$

$$\left(=\frac{\partial}{\partial t}-\sum_{|\nu|\leqq 2b}\overline{g}_\nu(\overline{x}')\left(\frac{\partial}{\partial \overline{x}}\right)^\nu\right),\ \text{where}\ \overline{g}_{ij}(\overline{x}')=\sum_{k=1}^{n}\frac{\partial F_k}{\partial \overline{x}_i}\frac{\partial F_k}{\partial \overline{x}_j}$$

$$(i, j=1,\cdots, n-1)$$

$$(\overline{g}_{ij}(\overline{x}')\in C^{s-1}),\ \overline{g}=\det(\overline{g}_{ij})\ \text{and}\ (\overline{g}^{ij})=(\overline{g}_{ij})^{-1}.$$

Then we have the fundamental solution $P(t, x, \xi)(t>0, x, \xi\in S)$ with the following properties.

i) $$P(t, x, \xi)=P(t, \xi, x),$$

ii) $$\mathcal{L}_{t,x}P(t, x, \xi)=0,$$

iii) $$\lim_{t\to+0}\int P(t, x, \xi)f(\xi)dS_\xi=f(x),\quad (f\text{: cont.})$$

iv) $$\int P(t-s, x, y)P(s-\tau, y, \xi)\,ds_y=P(t-\tau, x, \xi).$$

Using this, we can define the fractional power of $\mathcal{L}$ as follows.

$$K_\beta f(t, x)=\int^t d\tau\int K_\beta(t-\tau, x, \xi)f(\tau, \xi)\,dS_\xi\quad(\beta>0),$$

where $K_\beta(t, x, \xi)=\dfrac{t^{\beta-1}}{\Gamma(\beta)}P(t, x, \xi)$,

$$K_\beta f(t, x)=\mathcal{L}K_{\beta+1}f(t, x)\quad(-1<\beta\leqq 0),$$

and recurrently for all real β. It follows $\mathcal{L}K_\beta=K_{\beta-1}$, $K_\beta K_{\beta_1}=K_{\beta+\beta_1}$, $K_0=I$. Moreover we can prove

LEMMA 2.

i) Let $f(t, x)\in C_0^{\beta_1}(t, x)(x\in S)\ (\beta_1+2b\beta>0, \neq 1, 2,\cdots)$,

$$K_\beta f(t, x)\in C_0^{\beta_1+2b\beta}(t, x)\ \ (x\in S)$$

ii) Let $f(t, x; \tau_0, \xi_0)\in \widehat{C}_l^{\beta_1}(t-\tau_0, x-\xi_0)(x, \xi_0\in S)(\beta_1+2b\beta>0, \neq 1, 2,\cdots)$, if $l<n-1+2b$, then

$$(K_\beta f)(t, x; \tau_0, \xi_0)\in \widehat{C}^{\beta_1+2b\beta}_{l-2b\beta}(t-\tau_0, x-\xi_0)\ \ (x, \xi_0\in S)$$

Let us notice that if u satisfies the conditions

(3) $$K_{-\beta_j}B_ju=K_{-\beta_j}f_j\ \ (j=1,\cdots, b)\ \text{on}\ (0, T)\times S$$

and $$u\in C_0^{2b\beta_j+r_j+\delta}(t, x)(x\in\Omega)(\delta>0),$$

then u satisfies (2) also.

4. Construction of solution. Define the following functions for $(\sigma=(\sigma',\sigma_n)=(\sigma_1,\cdots,\sigma_{n-1},\sigma_n),\ \mathrm{Re}\,p=1,\ \xi\in\overline{V})$:

$$\overline{A}_0(\sigma;\tau,\xi)=\sum_{|\nu|=2b}\bar{a}_\nu(\tau,\xi)(i\sigma)^\nu\quad\Big(A=\sum_{|\nu|\leqq 2b}\bar{a}_\nu(t,x)\Big(\frac{\partial}{\partial\bar{x}}\Big)^\nu\Big),$$

$$\overline{B}_{0j}(\sigma;\tau,\xi)=\sum_{|\nu|=r_j}\bar{b}_{j\nu}(\tau,\xi)(i\sigma)^\nu\quad\Big(B_j=\sum_{|\nu|\leqq r_j}\bar{b}_{j\nu}(t,x)\Big(\frac{\partial}{\partial\bar{x}}\Big)^\nu\Big),$$

$\overline{A}_{0+}(p,\sigma',\sigma_n;\tau,\xi)$ and $\overline{R}(p,\sigma';\tau,\xi)$ are defined in the same way as in **1**, namely, they are obtained by replacing A_0, B_{0j} by $\overline{A}_0$, $\overline{B}_{0j}$ respectively. $\overline{R}_j(p,\sigma',\dot{x};\tau,\xi)$ are defined in the same way as $\overline{R}$, namely, by replacing $\overline{B}_{0j}(\sigma;\tau,\xi)$ by $e^{i\dot{x}\sigma_n}$. Define

$$\bar{g}_j(p,\sigma'\,\dot{x};\tau,\xi)=[p-\sum_{|\nu|=2b}\bar{g}_\nu(\xi)(i\sigma')^\nu]^{-\beta_j}\,\frac{\overline{R}_j(p,\sigma',\dot{x};\tau,\xi)}{\overline{R}(p,\sigma';\tau,\xi)}.$$

Then we put

$$\overline{G}_j(t,\bar{x}',\dot{x};\tau,\xi)=\frac{1}{(2\pi)^{ni}}\int_{\sigma'\in R^{n-1}}e^{i\bar{x}'\sigma'}d\sigma'\int_{\mathfrak{Re}\,p=1}e^{tp}\bar{g}_j(p,\sigma',\dot{x};\tau,\xi)dp\frac{1}{\sqrt{\bar{g}(\xi)}}$$

$(x\in\overline{U},\ \xi\in\overline{V})$; Next we put

$$G_j(t,x;\tau,\xi)=\textstyle\sum_I\bar{\alpha}(x)\overline{G}_j(t-\tau,\bar{x}(x)-\bar{\xi}(\xi);\tau,\xi)\bar{\alpha}(\xi),\ \text{and}$$

$$E_j(t,x;\tau,\xi)=G_j(t,x;\tau,\xi)-\int_\tau^t ds\int_\Omega Z(t,x;s,y)L_{s,y}G_j(s,y;\tau,\xi)\,dy,$$

$(t>\tau,\ x\in\Omega,\ \xi\in S)$, where $\{\bar{\alpha}(x)\}_I$ is a partition of unity of $\tilde{\tilde{S}}\subset\tilde{S}$, such that the element $\bar{\alpha}(x)$ is infinitely differentiable, its support is contained in $\overline{U}$, and $\sum_I\bar{\alpha}(x)^2=1$ on $\tilde{\tilde{S}}$, and $Z(t,x;\tau,\xi)$ is an elementary solution of (1). Then $\{E_j\}$ have the following properties well attached to our problem (1)–(2).

LEMMA 3.

i) $E_j(t,x;\tau,\xi)\in\widehat{C}^{2b+\gamma'}_{n+\varepsilon}(t-\tau,x-\xi)$ $(t>\tau,\ x\in\Omega,\ \xi\in S)$, and

$$\int_0^t d\tau\int E_j(t,x;\tau,\xi)f(\tau,\xi)\,dS_\xi\in C_0^{2b-1-\varepsilon+\delta'}(t,x)\quad(x\in\Omega)$$

for $f(t,x)\in C_0^\delta(t,x)$ $(x\in S)(\delta'<\delta\leqq\gamma)$.

ii) $\quad L_{t,x}E_j(t,x;\tau,\xi)=0\quad(t>\tau,\ x\in\Omega,\ \xi\in S)$.

iii) $\quad(K_{-\beta_i}B_iE_j)(t,x;\tau,\xi)=E^0_{ij}(t,x;\tau,\xi)-E^1_{ij}(t,x;\tau,\xi)$

$(t>\tau,\ x\in\tilde{S},\ \xi\in S)$, where

a) $$\int_0^t d\tau\int E^0_{ij}(t,x;\tau,\xi)f(\tau,\xi)\,dS_\xi\xrightarrow[\dot{x}\to 0]{}\begin{cases}f(t,x) & (i=j)\\ 0 & (i\neq j)\end{cases}$$

for $f(t,x)\in C_0^\delta(t,x)$ $(x\in S)$ $(\delta>0)$,

b) $\quad E^1_{ij}(t,x;\tau,\xi)\in\widehat{C}^{\gamma'}_{n-1+2b-r}(t-\tau,x-\xi)$ $(x\in\tilde{S},\ \xi\in S)$ $(\gamma'<\gamma)$.

Finally we put

$$\mathcal{E}_j(t,x;\tau,\xi)=E_j(t,x;\tau,\xi)+\textstyle\sum_i\int_\tau^t ds\int E_i(t,x;s,y)\Phi_{ij}(s,y;\tau,\xi)\,dS_y$$

$(t>\tau,\ x\in\Omega,\ \xi\in S)$, where

$$E_1(t,x;\tau,\xi)=(E^1_{ij}(t,x;\tau,\xi)),$$
$$E_{j+1}(t,x;\tau,\xi)=\int_\tau^t ds\int E_1(t,x;s,y)E_j(s,y;\tau,\xi)\,dS_y\quad(j=1,2,\cdots),$$

and

$$\Phi(t,x;\tau,\xi)=E_1(t,x;\tau,\xi)+E_2(t,x;\tau,\xi)+\cdots,$$

where $\Phi(t,x;\tau,\xi)\in\widehat{C}^{\gamma'}_{n-1+2b-r}(t-\tau,x-\xi)$ $(x,\xi\in S)$ $(\gamma'<\gamma)$. Then we have

COROLLARY. i), ii) are same as in Lemma 3.

iii) $$(K_{-\beta_1}B_i\mathcal{E}_j)(t,x;\tau,\xi)=E^0_{ij}(t,x;\tau,\xi)+\mathcal{E}_{ij}(t,x;\tau,\xi)$$

$(t>\tau,\ x\in\widetilde{S},\ \xi\in S)$, where

$$\mathcal{E}_{ij}(t,x;\tau,\xi)\in\widehat{C}^{\gamma'}_{n-1+2b-r}(t-\tau,x-\xi)\ (\gamma'<\gamma)\ \text{and}$$
$$\mathcal{E}_{ij}(t,x;\tau,\xi)\xrightarrow[\dot{x}\to 0]{}0.$$

Then we have

THEOREM 1. *Under the assumptions* i)–iv), *we can find a solution* u *of the problem* (1)–(2), *such that*

$$u(t,x)=\sum_j\int_0^t d\tau\int\mathcal{E}_j(t,x;\tau,\xi)K_{-\beta_j}f_j(\tau,\xi)\,dS_\xi,$$

and $u(t,x)\in C_0^{2b-1-\epsilon+\gamma'}(t,x)$ $(x\in\Omega)$ $(\gamma'<\gamma)$.

Define the Green function G of (1)–(2) as follows.

$G(t,x,\xi)=Z(t,x,\xi)-Z_c(t,x,\xi)$, where

$Z(t,x,\xi)=Z(t,x;0,\xi)$ and

$$Z_c(t,x,\xi)=\sum_j\int_0^t d\tau\int\mathcal{E}_j(t,x;\tau,y)K_{-\beta_j}B_jZ(\tau,y,\xi)\,dS_y.$$

Then we have

THEOREM 2. $(|k|\leqq 2b-1,\ t\in(0,T),\ x,\xi\in\Omega)$

i) (*in case of* $\max. r_j<2b-1$)

$$\left|\left(\frac{\partial}{\partial x}\right)^k Z_c(t,x,\xi)\right|\leqq Ct^{-\alpha(n+|k|)}e^{-\phi(t,x-\xi)-\phi(t,\dot{x})-\phi(t,\dot{\xi})}\ {}^{*)}$$

ii) (*in case of* $\max. r_j=2b-1$)

$$\left|\left(\frac{\partial}{\partial x}\right)^k Z_c(t,x,\xi)\right|\leqq C\dot{\xi}^{-\delta}t^{-\alpha(n+|k|-\delta)}e^{-\phi(t,x-\xi)-\phi(t,\dot{x})-\phi(t,\dot{\xi})}\quad(\delta>0).$$

References

[1] S. D. Eidelman: Dokl. Akad. (N. S.), nos. 142 (1962), 149, 150 (1963).

[2] V. P. Mihailov: Dokl. Akad. (N. S.), no. 129 (1959).

*) ϕ is defined by $\phi(t,x)=c\,|x/t^\alpha|^q$.