142. On a Construction of Annihilating Spaces

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1. Throughout this note we will use the notations and results in a previous paper: *Annihilators of von Neumann Algebras* (*Annihilating Spaces*), Bull. Kyushu Inst. Tech., (M. & N.S.), No. 10, pp. 25–39 (1963). We will quote it, whenever necessary, as [A. S.].

The trace-class (τc) of operators on a Hilbert space $\mathfrak{H}$ is a Banach space with the norm $\tau(A)$ for every $A\in(\tau c)$. We shall denote by $t(A)$ the trace on (τc) and by $(\tau c)_0$ a closed subspace $\{A \mid t(A)=0\}$ of (τc). And every operator of rank ≤ 1 on $\mathfrak{H}$ is represented by $f\otimes\bar{g}$ for $f, g\in\mathfrak{H}$. Hence we have $t(f\otimes\bar{g})=\langle f, g\rangle$.

Let $\mathfrak{T}$ be a closed subspace of $(\tau c)_0$ generated by operators of rank ≤ 1. If we put ${}^{\mathfrak{T}}\mathfrak{M}^f=\{g \mid f\otimes\bar{g}\in\mathfrak{T}\}$, then we can easily show that ${}^{\mathfrak{T}}\mathfrak{M}^f$ is a closed linear subspace of $\mathfrak{H}$ (cf. [A. S.], p. 30). Moreover, we put ${}^{\mathfrak{T}}\mathfrak{M}_f=\mathfrak{H}\ominus{}^{\mathfrak{T}}\mathfrak{M}^f$.

DEFINITION. *A closed subspace* $\mathfrak{T}$ *of* $(\tau c)_0$ *is called an annihilating space in a Hilbert space* $\mathfrak{H}$, *if it satisfies the following conditions*:

(1) $\mathfrak{T}$ *is generated by operators of rank* ≤ 1;

(2) $\mathfrak{T}$ *is self-adjoint, i.e., if* $A\in\mathfrak{T}$, *then* $A^*\in\mathfrak{T}$;

(3) *if* $g\in{}^{\mathfrak{T}}\mathfrak{M}_f$, *then* ${}^{\mathfrak{T}}\mathfrak{M}_g\subset{}^{\mathfrak{T}}\mathfrak{M}_f$.

In [A. S.], we characterized the annihilator $\mathfrak{R}^{\perp}$ of a von Neumann algebra $\mathfrak{R}$ as an annihilating space (cf. [A. S., Theorem 1]). Our purpose of this note is to construct an annihilating space concretely in a sense.

2. We shall state

LEMMA. *Let* $\mathfrak{R}$ *be a von Neumann algebra and let* $\mathfrak{R}'$ *be the commutant of* $\mathfrak{R}$. *Then a closed subspace* $\mathfrak{T}$ *of* $(\tau c)_0$ *generated by the set* $\{f\otimes\bar{g}, g\otimes\bar{f} \mid f\in E(\mathfrak{H}), g\in(I-E)(\mathfrak{H}), E\in\mathfrak{R}'\}$ *is an annihilating space.*

Proof. It is clear that $\mathfrak{T}$ satisfies the conditions (1), (2) of the above Definition.

Let $\mathfrak{M}_f^{\mathfrak{R}}$ be a closed linear subspace of $\mathfrak{H}$ generated by all the Xf $(X\in\mathfrak{R})$. Hence the projection $E_f^{\mathfrak{R}}$ on $\mathfrak{M}_f^{\mathfrak{R}}$ is an element of $\mathfrak{R}'$. Therefore, by definition of $\mathfrak{T}$, $\mathfrak{H}\ominus\mathfrak{M}_f^{\mathfrak{R}}\subset{}^{\mathfrak{T}}\mathfrak{M}^f$. Consequently, we have $\mathfrak{M}_f^{\mathfrak{R}}\supset{}^{\mathfrak{T}}\mathfrak{M}_f$ for every $f\in\mathfrak{H}$.

Now we shall show an inverse inclusion. If $f\in E(\mathfrak{H})$ and $g\in(I-E)(\mathfrak{H})$ for any $E\in\mathfrak{R}'$, then we have $Tf=TEf=ETf\in E(\mathfrak{H})$ for every $T\in\mathfrak{R}$. Therefore $t(T(f\otimes\bar{g}))=\langle Tf, g\rangle=0$ for every $T\in\mathfrak{R}$.

Hence by [A. S., Theorem 1],

$$\{f\otimes\bar{g},\ g\otimes\bar{f}\mid f\in E(\mathfrak{H}),\ g\in(I-E)(\mathfrak{H}),\ E\in\mathfrak{R}'\}\subset\mathfrak{R}^{\perp}.$$

Consequently, we have $\mathcal{T}\subset\mathfrak{R}^{\perp}$. Therefore ${}^{\mathcal{T}}\mathfrak{M}^{f}\subset{}^{\mathfrak{R}^{\perp}}\mathfrak{M}^{f}$ for every $f\in\mathfrak{H}$ and hence ${}^{\mathcal{T}}\mathfrak{M}_{f}\supset{}^{\mathfrak{R}^{\perp}}\mathfrak{M}_{f}$. But since we have ${}^{\mathfrak{R}^{\perp}}\mathfrak{M}_{f}=\mathfrak{M}_{f}^{\mathfrak{R}}$ (cf. [A. S. Lemma, 7]), we have ${}^{\mathcal{T}}\mathfrak{M}_{f}\supset\mathfrak{M}_{f}^{\mathfrak{R}}$ for every $f\in\mathfrak{H}$. Thus we have ${}^{\mathcal{T}}\mathfrak{M}_{f}=\mathfrak{M}_{f}^{\mathfrak{R}}$ for every $f\in\mathfrak{H}$. Therefore if $g\in{}^{\mathcal{T}}\mathfrak{M}_{f}=\mathfrak{M}_{f}^{\mathfrak{R}}$, then

$${}^{\mathcal{T}}\mathfrak{M}_{g}=\mathfrak{M}_{g}^{\mathfrak{R}}\subset\mathfrak{M}_{f}^{\mathfrak{R}}={}^{\mathcal{T}}\mathfrak{M}_{f}.$$

Hence $\mathcal{T}$ satisfies the condition (3) of the above Definition.

THEOREM. *Let $\mathcal{T}$ be the annihilating space given in the above Lemma. Then we have $\mathcal{T}=\mathfrak{R}^{\perp}$. Therefore, $\mathfrak{R}=\mathcal{T}^{\perp}$.*

Proof. In the proof of the above Lemma, we proved $\mathfrak{M}_{f}^{\mathfrak{R}}={}^{\mathcal{T}}\mathfrak{M}_{f}=\mathfrak{M}_{f}^{\mathcal{T}^{\perp}}$ for every $f\in\mathfrak{H}$ (cf. [A. S., Lemma 7]). But $\mathfrak{R}'$ is generated by all the projections $E_{f}^{\mathfrak{R}}$ and $(\mathcal{T}^{\perp})'$ is generated by all the projections $E_{f}^{\mathcal{T}^{\perp}}$. Therefore $\mathfrak{R}'=(\mathcal{T}^{\perp})'$ and thus $\mathfrak{R}=\mathcal{T}^{\perp}$. Consequently, $\mathcal{T}=\mathfrak{R}^{\perp}$.

REMARK. Let $\mathcal{T}$ be an annihilating space and let $\mathfrak{N}$ be a subset of $\mathfrak{H}$. We put ${}^{\mathcal{T}}\mathfrak{M}^{\mathfrak{N}}=\bigcap_{f\in\mathfrak{N}}{}^{\mathcal{T}}\mathfrak{M}^{f}$ and ${}^{\mathcal{T}}\mathfrak{M}_{\mathfrak{N}}=\mathfrak{H}\ominus{}^{\mathcal{T}}\mathfrak{M}^{\mathfrak{N}}$. And let ${}^{\mathcal{T}}E^{\mathfrak{N}}$, ${}^{\mathcal{T}}E_{\mathfrak{N}}$ be the projections on ${}^{\mathcal{T}}\mathfrak{M}^{\mathfrak{N}}$, ${}^{\mathcal{T}}\mathfrak{M}_{\mathfrak{N}}$ respectively. Then we can easily show that all the projections in $(\mathcal{T}^{\perp})'$ are the form ${}^{\mathcal{T}}E^{\mathfrak{N}}$, ${}^{\mathcal{T}}E_{\mathfrak{N}}$. Hence an annihilating space $\mathcal{T}$ not only is the annihilator of the von Neumann algebra $\mathcal{T}^{\perp}$ but determines the commutant $(\mathcal{T}^{\perp})'$ of $\mathcal{T}^{\perp}$ in the above mentioned sense.