138. E. R. Functional Integrals

By Hideo YAMAGATA

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1964)

§1. Introduction. In order to define Feynman integral exactly, the extension of the following spaces is required in functional integral: (1) the functional's domain $\mathfrak{H}$, (2) the set of integrable (not necessarily bounded) functionals defined on $\mathfrak{H}$. To perform the extension (1), outer Hilbert space [1] or neuclear space [2] is already constructed, because we cannot define even the completely additive Gauss measure in $\mathfrak{H}$. But the concrete meaning of these extended spaces are not yet obvious, and it is difficult to clarify this. Hence let's show here another more concrete and more delicate extension of $\mathfrak{H}$ by using E. R. integral without showing the relation between our extension and the neuclear space etc. The meaning of this extended space in Feynman integral is the increase of the considerable path in quantum field theory. It seems to us that this extended space gives the negative effect to the extension (2). For our purpose it needs to compensate this negative effect by the suitable use of both extensions (1) and (2). Furthermore, the extension (2) in Feynman integral permits us to use the more singular potential.

Here, using E. R. integral as the most general singular integral, the above extensions (1) and (2) for general functional integral is performed in the most wide meaning. Furthermore, the type of singularities in Feynman integral is investigated, and the possibility of the definition of E. R. Feynman integral constructed by the extensions (1) and (2) is shown. Recently, the equivalence between the primitive E. R. integral and A-integral by A. И. Kolmogoroff has been proved. A-integral has very simple form [8], but in order to construct the more wide organized extension of usual integral, E. R. form is more useful than A-integral.

§2. Definition of E. R. functional integral. Afterwards, we will use the same notations as one used in [5]. In this paragraph, we will extend the set of integrable functionals $\overline{C}$ defined in [5]. For its preliminaries, let's show an explanation about the concept of cylinder functional appearing in the usual definition of functional integral. Namely cylinder functional can be considered as the step functional which is constant in the set of $\xi(s)$ such that $P_j^{(N)}(\xi(s))=P_j^{(N)}(\xi^0(s))$ for any j, fixed D_N and fixed $\xi^0(s)$. The original $\overline{C}$ in [5] is the set of the functional f constructed by the convergent sequences of step functionals in $L^1(\mathfrak{H})$. Required improvements are the following two.

(1) The function $\xi(s)$ contained in the extended domain of functional is one approximated by the sequence of step functions in the more weak topology. (2) The functional $f(\xi)$ contained in the extension of $\overline{C}$ is one approximated by the sequence of cylinder functionals in the more weak topology. By the improvement (1), the domain of the integrand of this functional integral is extended, on the other hand the set of continuous functionals, consequently the set of the integrable functionals is restricted.

In Feynman integral, since the form of the functionals for fixed potential is determined, the suitable extension of $\mathfrak{H}$ is possible and needed. But for the treatment of the more singular case the extension of $\overline{C}$ is also required, because it extends the set of integrable functionals. Let's use these two extensions suitably, and construct the extended Feynman integral.

At the first step, let's investigate the approximation to $\xi(s)$ by the sequence of step functions. In [5], this sequence is converging to $\xi(s)$ in locally L^2 topology. Now, we use the convergence in ranked space instead of the convergence in L^2 topology.

For the extension of $\mathfrak{H}=L^2_{[0,1]}$, the convergent sequence in ranked space is used. Namely the neighbourhoods $V(F,\nu;\xi)=\{\eta(s);\eta(s)-\xi(s)\in V(F,\nu;0)\}$ are used. Here, $V(F,\nu;0)$ is the set of step functions $\eta(s)=p(s)+r(s)$ satisfied the conditions,

$$\text{(A) } r(s)=0 \text{ in } F, \text{ (B) } \int_0^1 |p(s)|ds<2^{-\nu}, \text{ (C) } \left|\int_0^1 r(s)ds\right|<2^{-\nu},$$

where $F\subset[0,1]$. Suppose that the selection of the sequence $\{V(F_n,\nu_n;\xi_n)\}$ with the following properties is possible. (a) $V(F_1,\nu_1;\xi_1)\supset V(F_2,\nu_2;\xi_2)\supset\cdots$ (b) $\xi(s)=\lim_{m\to\infty}\xi_m(s)$ (c) $\xi_{2n}=\xi_{2n+1}$, $\nu_{2n}<\nu_{2n+1}$ (d) $k\{\text{mes}([0,1]-F_{n+1})\}\geq\text{mes}([0,1]-F_n)$ for positive integer $k\geqq 2$ (e) there exists a function $\phi(n)$ which has the properties 1) $\phi(n)>0$, 2) $\lim_{n\to\infty}\phi(n)=0$, and 3) $\int_E|\xi_n(s)|ds\leq\phi(n)$ for $E\subset[0,1]$ with the measure $\text{mes}\,E<\text{mes}\{[0,1]-F_n\}$. Then this $\xi(s)$ is added to the domain of functional and the E. R. continuous functional is defined.

For the extension of $\mathfrak{H}=L^2_{(-\infty,+\infty)}$ the locally convergent sequence in ranked space is used. The limit function locally convergent sequence (by the above E. R. meaning) is contained in the domain of E. R. continuous functional in this case.

Here, we use the word "step function" to the step function with finite step. These step functions are used without the loss of generality. E. R. continuous functional is one with the following properties: (1) for the step function $\xi_m(s)$ with the fixed discrete point $\{s_0,s_1,\cdots,s_m\}$, $f(\xi_m(s))$ can be defined and becomes to a continuous function $F(x_1, x_2, \cdots, x_m)$ defined in the m dimensional Euclidean space E^m, where x_i $(i=1,2,\cdots,m)$ is the value of $\sqrt{s_i - s_{i-1}}\xi_m(s)$ in the interval $[s_{i-1}, s_i)$. (2) for the function $\xi(s)$ which is defined by $\lim_{m\to\infty} \xi_m(s)$ in the above ranked space, $f(\xi(s))$ can be defined and $f(\xi(s)) = \lim_{m\to\infty} f(\xi_m(s))$.

The integrable functional by this meaning can be represented by a sort of limit of this E. R. continuous functionals. But we do not discuss about this, because this is not main purpose of this paper.

At the second step, let's show the definition of E. R. integrable functional. The determination of F_n is the most difficult part in this definition. We denote by F_n the set of the function $\xi(s)$ which is contained in the complement of H_n and in the domain of the functional $f(\xi(s))$. Here, the set H_n is defined in the following.

For a division $D_n=\{s_0, s_1, \cdots, s_n\}$ of the real axis, $P_j^{(n)}(\xi(s))$ $(j=1,2, \cdots, n)$ can be defined (for $[s_{j-1}, s_j)$, $(s_{j-1}, s_j]$, $[s_{j-1}, s_j]$ or (s_{j-1}, s_j)). Let H_n denote the set of the function $\xi(s)$ such that $P_j^{(n)}(\xi(s))$ $(j=1, 2, \cdots, n)$ take the values in an open subset $O_j^{(n)}$ of the real axis. We may take all real axis as $O_j^{(n)}$. As the numbers related to H_n, $\varepsilon_j^{(n)}$ and $\tilde{\varepsilon}^{(n)}$ are given. Namely let $\varepsilon_j^{(n)}$ denote the measure of $O_j^{(n)}$, and $\tilde{\varepsilon}^{(n)}$ denote the sum of $(s_j - s_{j-1})$ such that $O_j^{(n)}$ is all real axis.

Let's take the open intervals as the intervals related to $\tilde{\varepsilon}^{(n)}$. Here, the following neighbourhood of cylinder functionals $V(F, \nu; f) = \{g(\xi(s)); g(\xi(s)) - f(\xi(s)) \in V(F, \nu; 0)\}$ is used. $V(F, \nu; 0)$ is the set of cylinder functionals $f(\xi(s)) = p(\xi(s)) + r(\xi(s))$ such that (A) $r(\xi(s)) = 0$ in F, (B) $\int |p(\xi(s))| d\xi(s) < 2^{-\nu}$, (C) $\left|\int r(\xi(s)) d\xi(s)\right| < 2^{-\nu}$, where F is the complement of some H_n. Since the cylinder functional $f(\xi(s))$ related to D_N is considered as the step functional taking a constant value in the set of $\xi(s)$ with the property $P_j^{(N)}(\xi(s)) = k_j$ (k_j fixed) for $j=1, \cdots, N$, this definition of neighbourhoods are possible.

Suppose that the selection of the sequence $\{V(F_n, \nu_n; f_n)\}$ with the following properties is possible.

(a) $V(F_1, \nu_1; f_1) \supseteq V(F_2, \nu_2; f_2) \supseteq \cdots$, where f_m are cylinder functionals for any m.

(b) $f(\xi(s)) = \lim_{m\to\infty} f_m(\xi(s))$ for fixed $\xi(s)$.

(c) $\bigcup_m D_m$ is the dense set of the real axis.

(d) $f_{2n} = f_{2n+1}$, $\nu_{2n} < \nu_{2n+1}$.

(e) $k \operatorname{mes} H_{n+1} \geqq \operatorname{mes} H_n$ for positive integer $k \geq 2$.

(f) There exists a function $\phi(n)$ which has the properties; 1) $\phi(n) > 0$, 2) $\lim_{n\to\infty} \phi(n) = 0$, 3) $\int_E |f_n(\xi(s))| d\xi(s) \leq \phi(n)$ for the set of function E with the measure $\operatorname{mes} E < \operatorname{mes} H_n$. Then $f(\xi(s))$ is the E. R. integrable functional and $\int f(\xi)(s))d\xi(s) \equiv \lim_{n\to\infty} \int f_n(\xi(s))\,d\xi(s) = \lim_{n\to\infty} \int \cdots \int_{\Pi_{j=1}^n[-l_n, l_n]-O_j^{(n)})} F_n(P_j^{(n)}(\xi(s)), \quad j=1,2,\cdots,n) \prod_{j=1}^n dP_j^{(n)}(\xi(s))$.

Here, $\lim_{n\to m} l_n = \infty$, $f_n(\xi(s)) = F_n(P_j^{(n)}(\xi(s)), \quad j=1,2,\cdots,n)$ and mes $(\prod_{j=1}^n([-l_n, l_n] \cap O_j^{(n)})) < \varepsilon_n$, where ε_n is a sequence of non-negative numbers tending to 0. The limit of the integral by Gauss measure is a sort of variation of this improper integral. Now, using this singular functional integral, let's define the extended Feynman integral.

§3. **E. R. Feynman integral** [2]. In this paragraph let's investigate the character of the singularities appearing in Feynman integral, and define the E. R. Feynman integral.

Let $\xi_n(s)$, $(\tilde{\xi}_n(s))$ denote the step functions converging to $\omega(s)$ $(\dot{\omega}(s))$ as $n\to\infty$ in ranked space.

At the first step, let's show the relation between $\xi_n(s)$ and $\tilde{\xi}_n(s)$. Suppose that the divided points using to construct $\xi_n(s)$ are denoted by $0=t_0<t_1<\cdots<t_n=t$. Then $\xi_n(s)=h_i$ for $s\in(t_{i-1}, t_i]$, $\tilde{\xi}_n(s)=h_0$ for $s\in(t_0, t_1]$, and $\tilde{\xi}_n(s)=(h_i-h_{i-1})/(t_i-t_{i-1})$ for $s\in(t_{i-1}, t_i)$, where h_0 is formed from the values of $i(1/2m)\Delta\varphi - iV\varphi$ in a fixed space point.

Using $E(\xi_n, \tilde{\xi}_n; t, m, V) \equiv \exp i \int_0^t [m\tilde{\xi}_n(s)^2/2 - V(\xi_n(s))]ds$ instead of $E(\omega, \dot{\omega}; t, m, V)$ n dimensional function $F(h_1, h_2, \cdots, h_n)$ is obtained. From the relation between Feynman integral and the initial problem of Schrödinger equation, we see that this relation between $\xi_n(s)$ and $\tilde{\xi}_n(s)$ is very suitable.

Lemma 1. *If $\dot{\omega}(s)$ (the element of $L^2_{[0,t]}$ is integrable and if $\omega(s)=\int_0^s \dot{\omega}(s)ds + C$ (C is the initial value $\omega(0)$), then the sequence of the above pair step functions $(\{\xi_n(s)\}, \{\tilde{\xi}_n(s)\})$ converging to $(\omega(s), \dot{\omega}(s))$ in $L^1 \times L^2$ can be constructed.* Here, we omit the proof of this lemma.

Let $\eta_n(s)$, $\tilde{\eta}_n(s)$, $\xi_n(s)$, and $\tilde{\xi}_n(s)$ be the step functions constructed by using the same set of divided points D_n.

Let D_n be the set of divided points such that $D_n \subset D_m$ for $n \le m$ and $\bigcup_n D_n$ is the dense set of considered real interval. Suppose that $(\{\xi_n(s)\}, \{\xi_n(s)\})$ converge to $(\omega, \dot{\omega})$ in ranked space.

Lemma 2. *If $|\xi_n(s)-\eta_n(s)| < \varepsilon_n$ and $|\tilde{\xi}_n(s)-\tilde{\eta}_n(s)| < \varepsilon_n$ $(\lim_{n\to\infty} \varepsilon_n = 0)$, then $\{\eta_n(s)\}$ and $\{\tilde{\eta}_n(s)\}$ are the sequence of step functions converging to $\omega(s)$ and $\dot{\omega}(s)$ by the meaning of ranked space.*

This lemma can be proved by using the character of the sequence of the neighbourhoods $\{V(F_n, \nu_n; f_n)\}$ in ranked space easily. Furthermore we can easily prove that if $(\{\xi_n(s)\}, \{\tilde{\xi}_n(s)\})$ tends to $(\omega, \dot{\omega})$ in $L^1 \times L^2$, then $(\{\eta_n(s)\}, \{\tilde{\eta}_n(s)\})$ also tends to $(\omega, \dot{\omega})$ in $L^1 \times L^2$.

If $V(s)$ is the bounded continuous potential, a functional $E(\omega, \dot{\omega}; t, m, V)$ used in Feynman integral is a continuous functional in $D^2_{L^2}$. Here, $D^2_{L^2} = \{f; (f, f') \in L^2 \times L^2\}$. But if $V(s)$ has singular points or if $V(s)$ is not bounded, we cannot define the functional $E(\omega, \dot{\omega}; t, m, V)$ even in the set of $\omega(s)$ defined in Lemma 1. Then we must select some suitable conditional convergent sequence $(\xi_n(s), \tilde{\xi}_n(s))$ and must define this functional for sufficiently wide $\omega(s)$ by using this sequence.

Next, let's investigate the effect of the square and the derivative. For an arbitrary measurable function $\dot{\omega}(s)$, we can find the density step function $\dot{\omega}(s)$ with the properties $\omega(s) = \text{E. R.} \int_0^s \dot{\omega}(s)\, ds + C$ in almost everywhere [7]. But, $\dot{\omega}(s)$ is not necessarily contained in $L^2{}_{[0,s]}$, because the results in Lemma 1 holds good. Corresponding to this $\dot{\omega}(s)$, there exists a bounded step function $\tilde{\xi}_n(s)$ and $\xi_n(s)$ such that $\lim_{n\to\infty} \tilde{\xi}_n(s) = \dot{\omega}(s)$ and $\lim_{n\to\infty} \int_0^s \tilde{\xi}_n(s)\, ds + C = \lim_{n\to\infty} \xi_n(s) + C = \omega(s)$ in almost everywhere. For these $\omega(s)$, we must give the following extension of functional used in Feynman integral: $E(\omega, \dot{\omega}; t, m, V) = \lim_{n\to\infty} E(\xi_n, \tilde{\xi}_n; t, m, V)$. Here, the meaning of $\lim_{n\to\infty}$ is to take an accumulated point of argument as its argument. If $\omega(s)$ has the suitable smoothness, we also use the initial condition $\dot{\omega}(0) = C'$. By the above consideration we can obtain the following

Theorem 1. *The singularity in $E(\omega, \dot{\omega}; t, m, V)$ is one of the following two type's singularities.*

(1) *the restriction of the domain derived from the singularities $V(s)$ (contained $s = \infty$),*

(2) *the restriction of the domain derived from the possibility of the derivation and the square integral.*

In order to define Feynman integral exactly, we must use the most general singular functional integral. E. R. functional integral is the most general and most constructive one. Using this mild integral, we can avoid the above singularities.

Now let's show them by comparing with our definition of E. R. functional integral. The first singularity can be avoided by the elimination of the open set $O_j^{(n)}$ with the measure $\varepsilon_j^{(n)} \neq 0$ defined in §2. The second singularity is the difficult one to avoid it. As we have already shown, the problem about integrability rises from this singularity. But, if the set of these paths has total measure zero in $\mathfrak{H}$ or in its extension, this singularity is not effective one. Here, this singularity which gives the essential effects to functional integral is eliminated by the process to determine the pair of step function's sequences $(\{\xi_n(s)\}, \{\tilde{\xi}_n(s)\})$ and to consider $\lim_{D_n\to\infty}$ by some determined rule except for the process to construct improper integral.

Furthermore even the other difficulties derived from $\lim_{D_n\to\infty}$ can be avoided by the elimination of the divided intervals related to $\tilde{\varepsilon}^{(n)}$ defined in §2.

From the above considerations we obtain the following

Theorem 2. *If the potential $V(s)$ has separated singular points, E. R. Feynman integral can be defined formally.*

According to these Theorems 1–2, we know that E. R. extension of functional integral is valuable for its application. Only the problem about the coefficients in Feynman integral is still remained [5].

References

[1] K. O. Friedricks and H. N. Shapiro, *et al.* Integration of Functionals. New York University Institute Mathematical Science (1957).

[2] И. М. Геиьфанд и Н. Я. Виленкин: Оьоьщение Функций и Деиствия Над Ними. IV. (1961).

[3] E. Nelson: Feynman integral and the Schrödinger equation. J. of Math. Phys., **5**, 3 (1964).

[4] K. Kunugi: Sur une generalisation de l'integrale. Fundamental and Applied Aspects of Mathematics, pp. 1–30 (1959).

[5] H. Yamagata: On the definition of functional integral. Proc. Japan Acad., **40**(7) (1964).

[6] I. Fujiwara: The correspondence principle 1. Proceedings of the Physical Seminar in Trondheim, no. 3 (1962).

[7] И.А. Виноградова: О Предсталении измеримой Функции неопределенным А-интегралом. Изв. Асад. Наук. Acad. Hryk. СССР., **26**, 581–604 (1961).

[8] A. И. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung. Springer, Berlin (1933).