136. On Quasi-Montel Spaces

By Kazuo KERA

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1964)

1. In the theory of topological linear spaces, many properties of Montel spaces have been studied. In this paper, we shall investigate the properties of the spaces having weaker condition than Montel spaces, called "quasi-Montel spaces". Throughout this paper, terminology and notation are the same as in [1], if nothing otherwise is mentioned. For example, a Montel space means a locally convex separative topological linear space in which every bounded subset is relatively compact, and which is not necessarily tonnelé.

Definition. *We say that a locally convex separative topological linear space E is a quasi-Montel space, if and only if each convex weakly compact ($\sigma(E, E')$-compact) subset is compact for the original topology of E.*

Obviously, each Montel space is a quasi-Montel space.

Theorem 1. *In order that a locally convex separative topological linear space E is a Montel space, it is necessary and sufficient that E is a semi-reflexive*[1] *quasi-Montel space.*

Proof. Necessity is trivial.

Sufficiency: For any closed bounded subset A of E, there is a convex closed bounded subset B containing A. From the semi-reflexivity, B is a weakly compact subset. So B is compact, because E is a quasi-Montel space. Therefore A is also compact.

Theorem 2.

(a) *A subspace of a quasi-Montel space is a quasi-Montel space.*

(b) *A product space of quasi-Montel spaces is a quasi-Montel space.*

(c) *A direct sum of quasi-Montel spaces is a quasi-Montel space.*

(d) *A strict inductive limit of countable many quasi-Montel spaces is a quasi-Montel space.*

Proof. (a) Let E be a quasi-Montel space and F be a subspace of E. Each convex weakly compact ($\sigma(F, F')$-compact) subset A of F is convex weakly compact ($\sigma(E, E')$-compact) in E. As E is a quasi-Montel space, A is a compact subset of E. Therefore A is also compact in F.

---

1) We say that a topological linear space E is semi-reflexive, if each continuous linear functional on E' is continuous for $\sigma(E', E)$-topology.

(b) Let A be a convex weakly compact subset of the product space $E=\prod_{\alpha} E_{\alpha}$. (Each E_{α} is a quasi-Montel space.) As each projection A_{α} of A on E_{α} is a convex weakly compact ($\sigma(E_{\alpha}, E'_{\alpha})$-compact) set in E_{α}, A_{α} is compact for the original topology of E_{α}. Hence $\prod_{\alpha} A_{\alpha}$ is compact. And A is closed in $\prod_{\alpha} A_{\alpha}$, so A is a compact subset of E.

(c) Let A be a convex weakly compact subset of the direct sum $E=\bigoplus_{\alpha} E_{\alpha}$. (Each E_{α} is a quasi-Montel space.) As A is bounded, there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $A\subset \bigoplus_{i=1}^{n} E_{\alpha_i}$. By the same steps as in the latter half of (b), it follows that A is a compact subset of E.

(d) Let A be a convex weakly compact subset of the strict inductive limit E of the sequence $\{E_i\}$. (Each E_i is a quasi-Montel space.) Then A is contained in some E_i, in which A is compact; and the conclusion follows.

2. Let (E, F) be a separative dual system of linear spaces E, F. Consider a locally convex separative topology τ on E whose dual space is F. Then according to the theorem of Mackey-Arens, we have

$$\sigma(E, F)<\tau<\tau(E, F) \text{ on } E.^{2)}$$

Now, we consider two topologies τ_1, τ_2 on E such that

$$\sigma(E, F)<\tau_1<\tau_2<\tau(E, F)$$

and suppose that τ_2 is a quasi-Montel topology on E. Then, obviously, τ_1 is also a quasi-Montel topology. As weak topology $\sigma(E, F)$ is a quasi-Montel topology, we have a problem if there is a maximum one (stronger than the others) among quasi-Montel topologies τ_λ such that

$$\sigma(E, F)<\tau_\lambda<\tau(E, F).$$

Lemma. *Let X be a set which is considered as topological spaces X_λ with various topologies τ_λ. Suppose there is a separative topology τ which is weaker than all τ_λ. If a subset A of X is compact for all τ_λ, then A is also compact for the weakest topology τ_0 among those which are stronger (in the sense $>$) than all τ_λ.*

Proof. We denote by Δ the diagonal subset of $\prod_{\lambda} X_\lambda$. Then X_0 is homeomorphic to Δ, so A_0 is homeomorphic to $\Delta\cap\prod_{\lambda} A_\lambda$. According to Tychonoff's theorem, $\prod_{\lambda} A_\lambda$ is compact in $\prod_{\lambda} X_\lambda$. The existence of the above mentioned topology τ implies that Δ is closed in $\prod_{\lambda} X_\lambda$. Hence $\Delta\cap\prod_{\lambda} A_\lambda$ is compact and so is A_0.

---

2) Here, for two topologies γ, δ on E, $\delta<\gamma$ means that the topology γ is stronger than δ or coincides with δ on E.

Theorem 3. *Consider a separative dual system* (E, F) *of linear spaces. Then there exists a maximum quasi-Montel topology in the system* T *of all quasi-Montel topologies* τ_λ *on* E *such that*

$$\sigma(E, F) < \tau_\lambda < \tau(E, F).$$

Proof. The system of $\sigma(E, F)$-compact convex sets is the same for any two topologies τ_1, τ_2 in T. As each topology τ_λ in T is a quasi-Montel topology, any weakly compact convex set A is compact for all τ_λ in T. Because the weak topology $\sigma(E, F)$ is separative, T has a maximum topology τ_0 and A is compact for τ_0. This topology is the desired one.

3. Next, we shall study the properties of quasi-Montel spaces by considering their dual spaces.

Theorem 4. *In order that a locally convex separative topological linear space* E *is a quasi-Montel space, it is necessary and sufficient that the topology of* E *is the topology of* $\mathfrak{S}$*-convergence for some family* $\mathfrak{S}$ *of* $\tau(E', E)$*-compact sets in* E'.

Proof. (1) We suppose that E is a quasi-Montel space. Let B be an equi-continuous subset of E'. The topology of uniform convergence on each precompact set of E coincides with $\sigma(E', E)$ on B. As E is a quasi-Montel space, $\tau(E', E)$ coincides with $\sigma(E', E)$ on B. Then B is a relative $\tau(E', E)$-compact set.

(2) Conversely, we suppose that the topology of E is the topology of $\mathfrak{S}$-convergence for some family $\mathfrak{S}$ of $\tau(E', E)$-compact sets in E'. Let A be a convex weakly compact subset of E, then A is an equi-continuous subset of E as dual space of $(E')_{\tau(E', E)}$. Therefore, on A, the topology of $\mathfrak{S}$-convergence coincides with $\sigma(E, E')$. So A is compact for the original topology of E.

Remark. In Theorem 4, we can assume without loss of generality that the sets in $\mathfrak{S}$ are convex.

Corollary 1. *The maximum quasi-Montel topology on* E *is the one of* $\mathfrak{S}$*-convergence for the family* $\mathfrak{S}$ *of all* $\tau(E', E)$*-compact convex subsets in* E'.

Corollary 2. *We suppose that the topology of* E *is Mackey's topology* $\tau(E, E')$ *and* E *is a quasi-Montel space. Then* E' *is a quasi-Montel space for Mackey's topology* $\tau(E', E)$.

Corollary 3. *If the maximum quasi-Montel topology on* E *coincides with the weak topology* $\sigma(E, E')$, *each convex* $\tau(E', E)$*-compact subset of* E' *is contained in a linear subspace of finite dimensions.*

4. In a locally convex separative topological linear space over the real field, each continuous linear functional on E has the maximum value on a compact set. The converse is not always true. We shall consider the converse in a quasi-Montel space.

Lemma. *Let* A *be a weakly closed subset of a locally convex* *separative topological linear space E over the real field. If each continuous linear functional f on E has the maximum (and so the minimum) value on A, A is weakly compact in E.*

Proof. We denote by a_f and b_f, the minimum value and the maximum value of f on A respectively. Then the set A with the weak topology is topologically imbedded into the product of closed intervals $\prod_f [a_f, b_f]$, and the image is closed. Hence A is compact for the weak topology.

According to this lemma, it is easy to prove the following

Theorem 5. *Let E be a locally convex separative topological linear space over the real field. Then E is a quasi-Montel space, if and only if each convex closed subset of E on which any element of E' has the maximum value, is compact.*

5. Here we shall mention some examples and counter-examples on quasi-Montel spaces.

(A) The Banach space l^1 of all absolutely summable sequences, is a quasi-Montel space,[3)] but not a Montel space.

(B) (1) The Banach space $l^p\,(1<p<\infty)$ of p-th summable sequences, is reflexive, but not a Montel space. It is not a quasi-Montel space on account of Theorem 1.

(2) We consider the Banach space c_0 of all sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n=0$.

We take the sequence $\{X_n \mid X_n=(\delta_{nm})_{m=1,2,\dots}\}$ in c_0. As the dual space of c_0 is l^1, $\{X_n\}$ converges to 0 for the weak topology. But $\{X_n\}$ does not converge in c_0. Then c_0 is not a quasi-Montel space, because the convex closure of $\{X_n\}$ has the same property.

(3) We consider the Banach space l^∞ of all bounded sequences, and the Banach space c of all convergent sequences. Both contain c_0 as their subspace. So they are not quasi-Montel spaces by Theorem 2, (a).

Remark 1. The spaces c, c_0, $l^p\,(1<p<\infty)$ are separable Banach spaces. So each of them is topologically isomorphic to some quotient space of l^1.[4)] The fact that l^1 is a quasi-Montel space, but they are not quasi-Montel spaces, shows that the quotient space of a quasi-Montel space is not always a quasi-Montel space.

Remark 2. As the dual space of l^1 is l^∞, the dual space of a quasi-Montel space is not always a quasi-Montel space.

Remark 3. The usual topology of l^2 is Mackey's topology and l^2 is not a quasi-Montel space for this topology. The maximum quasi-Montel topology of l^2 is the topology of the uniform conver-

3) [2] p. 284, (3).
4) [2] p. 283, (1).

gence on each convex compact set of $(l^2)'=l^2$.

Remark 4. The usual topology of l^1 is Mackey's topology and it is the maximum quasi-Montel topology.

References

[1] A. Grothendieck: Espaces vectoriels topologiques. Departamento de Matemática da Universidade de São Paulo (1954).
[2] G. Köthe: Topologische lineare Räume I. Springer (1960).