

## 152. On the Perturbation of the Continuous Spectrum of the Dirac Operator

By Kiyoshi MOCHIZUKI

Department of Mathematics, Kyoto University  
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**1. Introduction.** In this note we are concerned with the Dirac operator arising in the relativistic quantum theory. Our purpose is to show the unitary equivalence between the free Dirac operator and the continuous part of the perturbed operator under certain conditions on the perturbation. The similar problems concerned with the Schroedinger operator have been studied by several authors. Among them the work of Friedrichs [3] is of interest from the viewpoint of time independent construction of the wave operator. Recently this so-called stationary method was developed by Faddief [2]. The method used here is essentially the same to that of Faddief.<sup>1)</sup>

**2. Operators in the momentum space.** Let  $\mathcal{H}$  be the set of four-component vector-valued functions  $f(\xi) \equiv (f_1(\xi), \dots, f_4(\xi))$  defined on  $E_3$  such that  $f_k(\xi) \in L^2(E_3)$  ( $k=1, 2, 3, 4$ ).  $\mathcal{H}$  forms a Hilbert space with respect to the inner product

$$\langle f, g \rangle \equiv \int_{E_3} f(\xi) \cdot \overline{g(\xi)} d\xi = \int_{E_3} \sum_{k=1}^4 f_k(\xi) \overline{g_k(\xi)} d\xi.$$

The free Dirac operator is given, as a multiplicative operator in  $\mathcal{H}$ , by

$$(2.1) \quad L_0 f \equiv L_0(\xi) f(\xi) = \left\{ \sum_{k=1}^3 A_k \xi_k + A_4 \right\} f(\xi), \quad \xi = (\xi_1, \xi_2, \xi_3) \in E_3,$$

where  $A_k$  ( $k=1, 2, 3, 4$ ) are constant matrices which satisfy the relations  $A_j A_k + A_k A_j = 2\delta_{jk} I$ . It turns out that  $L_0$  is a self-adjoint operator with the domain  $\mathcal{D} \equiv \{f \in \mathcal{H}; (1 + |\xi|) f_k(\xi) \in L^2(E_3), k=1, 2, 3, 4\}$ . Moreover  $L_0$  has no eigen-value and the continuous spectrum consists of all real  $\lambda$  such that  $|\lambda| \geq 1$ .

Let  $V$  be an integral operator

$$(2.2) \quad Vf = \int_{E_3} V(\xi - \eta) f(\eta) d\eta.$$

Throughout what follows we assume the matrix-valued function  $V(\xi) \equiv [v_{jk}(\xi)]_{j,k=1,2,3,4}$  satisfies the following conditions.

$$(C.1) \quad \begin{aligned} v_{jk}(\xi) &= \overline{v_{kj}(-\xi)}; & |v_{jk}(\xi)| &\leq \text{const} (1 + |\xi|)^{-2-\epsilon_0}; \\ |v_{jk}(\xi) - v_{jk}(\xi + h)| &\leq \text{const} (1 + |\xi|)^{-2-\epsilon_0} |h|^{\nu_0}, \end{aligned}$$

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1) Our problems are reduced to that of singular integrals with respect to the functions with Hoelder-continuity.

where  $\varepsilon_0 > 0$ ,  $\gamma_0 > \frac{1}{2}$  are constants independent of  $\xi$  and  $h$  ( $|h| \leq 1$ ).

(C.2) The Fourier transform  $\hat{V}(x)$  of  $V(\xi)$  is hermitian and  $\hat{v}_{jk}(x) \in C^1(E_3)$ ; there exists a positive constant  $R$  such that  $\hat{v}_{11}(x) = \dots = \hat{v}_{44}(x) \equiv \hat{v}(x)$  and for  $j \neq k$   $\hat{v}_{jk}(x) = 0$ , when  $|x| \geq R$ ;  $|\hat{v}(x)| + \sum_{k=1}^3 \left| \frac{\partial}{\partial x_k} \hat{v}(x) \right| \leq \text{const } |x|^{-2-\varepsilon_1}$ ,  $\varepsilon_1 > 0$ .

We introduce the perturbed Dirac operator  $L$  by setting

$$(2.3) \quad L = L_0 + V.$$

Under the assumption (C.2),  $L$  has the following properties:

*L is a self-adjoint operator with the domain  $\mathcal{D}$ . On the real segment  $[-1, 1]$ , there appears only a finite number of eigen-values of finite multiplicity. All real  $\lambda$  such that  $|\lambda| \geq 1$  are the points of the continuous spectrum.<sup>2)</sup>*

In the following we denote by  $E_0(\lambda)$ ,  $E(\lambda)$ , and  $R_0(z)$ ,  $R(z)$  ( $\text{Im } z \neq 0$ ) the respective resolutions of the identity and resolvents of  $L_0$  and  $L$ .

### 3. Projective operators associated with the free Dirac operator.

Let  $\mathcal{D}_{\delta, \gamma}$  ( $\delta, \gamma > 0$ ) be the set of all  $f \in \mathcal{H}$  which satisfy the following estimate:

$$\|f\|_{\delta, \gamma} \equiv \sup_{\varepsilon, |h| \leq 1} (1 + |\xi|)^\delta \sum_{k=1}^4 \{ |f_k(\xi)| + |h|^{-\gamma} |f_k(\xi) - f_k(\xi + h)| \} < +\infty.$$

If we note

$$(3.1) \quad R_0(z)f = (|\xi|^2 + 1 - z^2)^{-1} \{ L_0(\xi) + zI \} f(\xi), \quad f \in \mathcal{H},$$

then we can prove easily

**Lemma 3.1.** *Let  $f, g \in \mathcal{D}_{2+\varepsilon, \gamma}$  ( $\varepsilon > 0$ ,  $\gamma > \frac{1}{2}$ ), then  $\langle E_0(\lambda)f, g \rangle$  is continuously differentiable for  $|\lambda| \geq 1$  and*

$$(3.2) \quad \frac{d}{d\lambda} \langle E_0(\lambda)f, g \rangle = \frac{\lambda}{2|\lambda|} \left[ |\xi| \int_{\Omega} \{ L_0(\xi) + \lambda I \} f(\xi) \cdot \overline{g(\xi)} d\Omega_\xi \right]_{|\xi|^2 + 1 = \lambda^2},$$

where  $\int_{\Omega} d\Omega_\xi$  means the integration over the unit sphere.

Now we define the projective operators  $P_\alpha(\mu)$  ( $\alpha = p, n$ ) by  $P_p(\mu) = \int_1^\mu dE_0(\lambda)$ ,  $P_n(\mu) = \int_{-\mu}^{-1} dE_0(\lambda)$ . It follows, from the above lemma, that

$$(3.3) \quad P_p(\mu)f = \frac{1}{2} \begin{cases} \frac{L_0(\xi)}{\sqrt{|\xi|^2 + 1}} + I \} f(\xi), & \text{when } |\xi| \leq \sqrt{\mu^2 - 1} \\ = 0, & \text{when } |\xi| > \sqrt{\mu^2 - 1}, \end{cases}$$

2) For their purpose we use the operator in the energy space which is given, as the Fourier transform of (2.3), by

$$\hat{L}\hat{f} \equiv \hat{L}_0\hat{f} + \hat{V}\hat{f} = \left\{ \sum_{k=1}^3 A_k \frac{1}{i} \frac{\partial}{\partial x_k} + A_4 \right\} \hat{f} + \hat{V}(x)\hat{f}.$$

They follow easily from the results of Birman [1] and Kato [4]. Note that  $\hat{L}_0\hat{L}_0\hat{f} = (-\mathcal{A} + 1)\hat{f}$ .

$$(3.4) \quad P_n(\mu)f = \frac{1}{2} \left\{ \frac{-L_0(\xi)}{\sqrt{|\xi|^2+1}} + I \right\} f(\xi), \quad \text{when } |\xi| \leq \sqrt{\mu^2-1}$$

$$= 0, \quad \text{when } |\xi| > \sqrt{\mu^2-1}.$$

These formulas hold for all  $f \in \mathcal{H}$ . We denote the rights by  $P_n(\xi, \mu)f(\xi)$ .

4. **Some lemmas.** Let  $T(z)$  be the operator satisfying the relation  $T(z)R_0(z) = VR(z)$ . In this section we give some lemmas connected with the operator  $T(z)$ . We remark firstly that with the aid of  $T(z)$ ,  $R(z)$  can be expressed in the form:

$$(4.1) \quad R(z) = R_0(z) - R_0(z)T(z)R_0(z).$$

$T(z)$  turns out to be an integral operator, and the kernel  $T(\xi, \eta; z) \equiv [t_{jk}(\xi, \eta; z)]_{j,k=1,2,3,4}$  is obtained from the integral equation

$$(4.2) \quad T(\xi, \eta; z) = V(\xi - \eta) - \int_{\mathcal{D}_s} \frac{V(\xi - \zeta)\{L_0(\zeta) + zI\}T(\zeta, \eta; z)}{|\zeta|^2 + 1 - z^2} d\zeta.$$

To solve this equation, we follow the similar way to that of Faddeev [2].

**Lemma 4.1.** *Let  $\Pi$  be the complex plane with slits along the real line such that  $|\lambda| \geq 1$ . If  $z \in \Pi$  is not the point of eigen-value of  $L$ , then the equation (4.2) is solved uniquely, and for arbitrary positive number  $\rho$ , if  $1 \leq |Re z| \leq \rho$ , then each component of the solution  $T(\xi, \eta; z)$  satisfies the inequalities*

$$1^\circ) \quad |t_{jk}(\xi, \eta; z)| \leq C(\rho)(1 + |\xi - \eta|)^{-2-\varepsilon},$$

$$2^\circ) \quad |t_{jk}(\xi, \eta; z) - t_{jk}(\xi + h, \eta + h'; z + \Delta)|$$

$$\leq C(\rho)(1 + |\xi - \eta|)^{-2-\varepsilon}\{|h|^r + |h'|^r + |\Delta|^r\},$$

where  $0 < \varepsilon < \varepsilon_0$ ,  $\frac{1}{2} < r < r_0$ , and  $C(\rho)$  is a constant depending on  $\rho$ .

Now we recall the set  $\mathcal{D}_{\delta,r}$ . Since  $\|\cdot\|_{\delta,r}$  defines a norm on  $\mathcal{D}_{\delta,r}$ , it is regarded to be a Banach space with respect to this norm. We denote this by  $\mathcal{M}_{\delta,r}$ . Note that  $\|f\|_{\delta,r} \leq C\|f\|_{\delta',r}$  when  $\delta < \delta'$ , and

$$(4.3) \quad \|f\|_{\mathcal{H}} \equiv \sqrt{\langle f, f \rangle} \leq C\|f\|_{1+\beta,r}, \quad \beta > \frac{1}{2}.$$

Let us consider the operator  $K^*(\lambda + i\theta) \equiv T(\lambda + i\theta)R_0(\lambda + i\theta)$  ( $\lambda, \theta$ : real). Using the inequalities of the above lemma, we can verify that when  $|\lambda| < \rho$ ,  $K^*(\lambda + i\theta)$  is uniformly bounded for  $\theta$  in  $\mathcal{M}_{2+\varepsilon,r}$  and satisfies

$$(4.4) \quad \|K^*(\lambda + i\theta)f - K^*(\lambda + i\theta')f\|_{2+\varepsilon,r} \leq C(\rho)\|f\|_{2+\varepsilon,r}|\theta - \theta'|^r.$$

Hence we have

**Lemma 4.2.** *As  $\theta \rightarrow \pm 0$ ,  $K^*(\lambda + i\theta)f$  converges in  $\mathcal{M}_{2+\varepsilon,r}$ .*

Moreover, if we denote the limits by  $K^{(\pm)*}(\lambda)f$ , then by (4.3) we have

**Lemma 4.3.** *As  $\theta \rightarrow \pm 0$ ,  $K^*(\lambda + i\theta)f$  converges to  $K^{(\pm)*}(\lambda)f$  in  $\mathcal{H}$ .*

If we note that these convergences are locally uniform for  $\lambda$ , then we get immediately the following:

**Lemma 4.4.** *Let  $H_p^{(\pm)*}(\mu)f = [P_p(\mu)K^{(\pm)*}(\lambda)f](\xi)|_{\lambda = \sqrt{|\xi|^2+1}}$ , and let  $H_p^{(\pm)}(\mu)$  be the adjoint operators of  $H_p^{(\pm)*}(\mu)$ , then when  $f \in \mathcal{D}_{2+\varepsilon,r}$ ,*

$$(4.5) \quad H_p^{(\pm)*}(\mu)f = \text{l.i.m.}_{\theta \rightarrow +0} P_p(\xi, \mu) \times \\ \times \int_{\mathcal{E}_3} \frac{T(\xi, \eta: \sqrt{|\xi|^2 + 1} \pm i\theta) \{L_0(\eta) + (\sqrt{|\xi|^2 + 1} \pm i\theta)I\} f(\eta)}{|\eta|^2 + 1 - (\sqrt{|\xi|^2 + 1} \pm i\theta)^2} d\eta,$$

and when  $f \in \mathcal{D}_0(\mu) \equiv \{f \in \mathcal{A}; f_k(\xi) \in C_0^\infty(|\xi|^2 \leq \mu^2 - 1) \quad k=1, 2, 3, 4\}$ ,

$$(4.6) \quad H_p^{(\pm)*}(\mu)f = \text{l.i.m.}_{\theta \rightarrow +0} \\ \int_{\mathcal{E}_3} \frac{\{L_0(\xi) + (\sqrt{|\eta|^2 + 1} \mp i\theta)I\} T(\xi, \eta: \sqrt{|\eta|^2 + 1} \mp i\theta) P_p(\eta, \mu) f(\eta)}{|\xi|^2 + 1 - (\sqrt{|\eta|^2 + 1} \mp i\theta)^2} d\eta.$$

The similarly defined operators  $H_n^{(\pm)*}(\mu)$  and  $H_n^{(\pm)}(\mu)$  satisfy the same expressions.

**5. Expansion theorem.** Firstly we show the absolute continuity of  $E(\lambda)$ , namely

**Lemma 5.1.** *When  $f, g \in \mathcal{D}_{2+,r}$ ,  $\langle E(\lambda)f, g \rangle$  is continuously differentiable for  $|\lambda| \geq 1$  and,*

$$(5.1) \quad \frac{d}{d\lambda} \langle E(\lambda)f, g \rangle = \frac{\lambda}{2|\lambda|} \left[ |\xi| \int_a \{L_0(\xi) + \lambda I\} \{I - K^{(\pm)*}(\lambda)\} f(\xi) \cdot \right. \\ \left. \cdot \overline{\{I - K^{(\pm)*}(\lambda)\} g(\xi)} d\Omega_\xi \right]_{|\xi|^2 + 1 = \lambda^2}.$$

**Proof.** Taking into account of the second resolvent equation and the equality (4.1), we get

$$R(\lambda + i\theta) - R(\lambda - i\theta) = \{I - R_0(\lambda \pm i\theta)T(\lambda \pm i\theta)\} \times \\ \times \{R_0(\lambda + i\theta) - R_0(\lambda - i\theta)\} \{I - T(\lambda \mp i\theta)R_0(\lambda \mp i\theta)\}.$$

Therefore by virtue of Lemma 4.2, the relation

$$\langle \{R(\lambda + i\theta) - R(\lambda - i\theta)\} f, g \rangle = \\ = \langle \{R_0(\lambda + i\theta) - R_0(\lambda - i\theta)\} \{I - K^{(\pm)*}(\lambda)\} f, \{I - K^{(\pm)*}(\lambda)\} g \rangle + o(\theta)$$

holds for  $f, g \in \mathcal{D}_{2+,r}$ . Lemma 3.1. leads it to (5.1).

Let  $Q_p(\mu) = \int_1^\mu dE(\lambda)$  and  $Q_n(\mu) = \int_{-\mu}^{-1} dE(\lambda)$ . Then from the above

lemma, we get

$$(5.2) \quad \langle Q_\alpha(\mu)f, g \rangle = \langle \{P_\alpha(\mu) - H_\alpha^{(\pm)*}(\mu)\} f, \{P_\alpha(\mu) - H_\alpha^{(\pm)*}(\mu)\} g \rangle.$$

Here we introduce the operators  $W_\alpha^{(\pm)*}(\mu)$  by setting

$$(5.3) \quad W_\alpha^{(\pm)*}(\mu) = P_\alpha(\mu) - H_\alpha^{(\pm)*}(\mu), \quad \alpha = p, n.$$

From (5.2) we see that  $W_\alpha^{(\pm)*}(\mu)$  are bounded operators. Hence they are extended all over  $\mathcal{A}$ . Moreover the strong limits  $W_\alpha^{(\pm)*} \equiv s\text{-lim}_{\mu \rightarrow \infty} W_\alpha^{(\pm)*}(\mu)$  exist and are also bounded.

Let  $f_\alpha^{(\pm)} = W_\alpha^{(\pm)*} f$ , let  $\lambda_\nu$  and  $\varphi^{(\nu)}(\xi)$  ( $\nu=1, 2, \dots, N$ ) be the corresponding eigen-values and eigen-vectors of  $L$ , and let  $f^{(\nu)} = \langle f, \varphi^{(\nu)} \rangle$ . Then we have the following diagonal representation of  $L$ .

**Theorem 5.1.** *When  $f \in \mathcal{D}$  and  $g \in \mathcal{A}$ ,*

$$(5.4) \quad \langle Lf, g \rangle = \sum_{\nu=1}^N \lambda_\nu f^{(\nu)} \overline{g^{(\nu)}} + \int_{\mathcal{E}_3} \sqrt{|\xi|^2 + 1} f_p^{(\pm)}(\xi) \cdot \overline{g_p^{(\pm)}(\xi)} d\xi -$$

$$-\int_{E_3} \sqrt{|\xi|^2+1} f_n^{(\pm)}(\xi) \cdot \overline{g_n^{(\pm)}(\xi)} d\xi.$$

**6. Unitary equivalence.** Let  $W_\alpha^{(\pm)}(\mu)$  be the adjoint operators of  $W_\alpha^{(\pm)*}(\mu)$ . They are also bounded operators, and are given, if we restrict the domain to  $\mathcal{D}_0(\mu)$ , by

$$(6.1) \quad W_\alpha^{(\pm)}(\mu) = P_\alpha(\mu) - H_\alpha^{(\pm)}(\mu), \quad \alpha = p, n.$$

$W_\alpha^{(\pm)}(\mu)$  converge weakly as  $\mu \rightarrow \infty$ , because  $W_\alpha^{(\pm)*}(\mu)$  have the strong limits. Let  $W_\alpha^{(\pm)} = w\text{-}\lim_{\mu \rightarrow \infty} W_\alpha^{(\pm)}(\mu)$ . Now we are ready to prove

**Theorem 6.1.**  $W_\alpha^{(\pm)}$  are partially isometric operators from  $P_\alpha(\infty)\mathcal{H}$  onto  $Q_\alpha(\infty)\mathcal{H}$ , and satisfy the following relations:

$$(6.2) \quad L W_\alpha^{(\pm)} = W_\alpha^{(\pm)} L_0, \quad \alpha = p, n.$$

For this it is sufficient to prove

**Lemma 6.1.** For arbitrarily given constant  $\mu \geq 1$ ,

$$1^\circ) W_\alpha^{(\pm)}(\mu) W_\alpha^{(\pm)*}(\mu) = Q_\alpha(\mu), \quad 2^\circ) W_\alpha^{(\pm)*}(\mu) W_\alpha^{(\pm)}(\mu) = P_\alpha(\mu), \quad \alpha = p, n.$$

**Proof.**  $1^\circ)$  is obvious from (5.2). For  $2^\circ)$ , we only show when  $\alpha = p$ . We can restrict the domains of  $W_p^{(\pm)*}(\mu)$  and  $W_p^{(\pm)}(\mu)$  to  $\mathcal{D}_{2+\epsilon, r}$  and  $\mathcal{D}_0(\mu)$  respectively. Then the following relations hold.

$$W_p^{(\pm)*}(\mu) W_p^{(\pm)}(\mu) = P_p(\mu) - \{P_p(\mu) H_p^{(\pm)}(\mu) + H_p^{(\pm)*}(\mu) P_p(\mu) + H_p^{(\pm)*}(\mu) H_p^{(\pm)}(\mu)\}.$$

By virtue of Lemma 4.4,  $H_p^{(\pm)*}(\mu) H_p^{(\pm)}(\mu)$  are obtained as the weak limits of the integral operators with the kernels

$$J_1^{(\pm)}(\xi, \eta; \mu, \theta) = P_p(\xi, \mu) \int_{E_3} \frac{T(\xi, \zeta: \sqrt{|\xi|^2+1} \pm i\theta) \{L_0(\zeta) + (\sqrt{|\xi|^2+1} \pm i\theta) I\}}{|\zeta|^2+1 - (\sqrt{|\xi|^2+1} \pm i\theta)^2} \times \\ \times \frac{\{L_0(\zeta) + (\sqrt{|\eta|^2+1} \mp i\theta) I\} T(\zeta, \eta: \sqrt{|\eta|^2+1} \mp i\theta)}{|\zeta|^2+1 - (\sqrt{|\eta|^2+1} \mp i\theta)^2} d\zeta P_p(\eta, \mu).$$

If we note the equality

$$T(z_1) - T(z_2) + (z_1 - z_2) T(z_1) R_0(z_1) R_0(z_2) T(z_2) = 0, \quad z_1, z_2 \in \Pi,$$

we get

$$J_1^{(\pm)}(\xi, \eta; \mu, \theta) = \frac{P_p(\xi, \mu) \{T(\xi, \eta: \sqrt{|\xi|^2+1} \pm i\theta) - T(\xi, \eta: \sqrt{|\eta|^2+1} \mp i\theta)\} P_p(\eta, \mu)}{-\sqrt{|\xi|^2+1} + \sqrt{|\eta|^2+1} \mp 2i\theta}.$$

On the other hand, since the relations

$$\{L_0(\eta) + (\sqrt{|\xi|^2+1} \pm i\theta) I\} P_p(\eta, \mu) = (\sqrt{|\xi|^2+1} + \sqrt{|\eta|^2+1} \pm i\theta) P_p(\eta, \mu),$$

$$P_p(\xi, \mu) \{L_0(\xi) + (\sqrt{|\eta|^2+1} \mp i\theta) I\} = (\sqrt{|\xi|^2+1} + \sqrt{|\eta|^2+1} \mp i\theta) P_p(\xi, \mu)$$

hold, it turns out that  $P_p(\mu) H_p^{(\pm)}(\mu) + H_p^{(\pm)*}(\mu) P_p(\mu)$  are the strong limits of the integral operators with the kernels

$$J_2^{(\pm)}(\xi, \eta; \mu, \theta) = \frac{P_p(\xi, \mu) \{T(\xi, \eta: \sqrt{|\xi|^2+1} \pm i\theta) - T(\xi, \eta: \sqrt{|\eta|^2+1} \mp i\theta)\} P_p(\eta, \mu)}{-\sqrt{|\xi|^2+1} + \sqrt{|\eta|^2+1} \mp i\theta}.$$

Therefore

$$\langle H_p^{(\pm)*}(\mu) H_p^{(\pm)}(\mu) f, g \rangle - \langle \{P_p(\mu) H_p^{(\pm)}(\mu) + H_p^{(\pm)*}(\mu) P_p(\mu)\} f, g \rangle = \\ = \lim_{\theta \rightarrow +0} \int_{E_3} \overline{g(\xi)} \cdot \int_{E_3} \{J_1^{(\pm)}(\xi, \eta; \mu, \theta) - J_2^{(\pm)}(\xi, \eta; \mu, \theta)\} f(\eta) d\eta d\xi = 0,$$

when  $f \in \mathcal{D}_0(\mu)$  and  $g \in \mathcal{H}$ . This proves 2°).

**Final remark.** Put  $S_\alpha = W_\alpha^{(+)*} W_\alpha^{(-)}$ , and  $S = S_p \oplus S_n$ , then  $S$  is shown to be a unitary operator. If we make forward the discussion carefully,  $S$  will be shown to have the properties of the so-called  $S$ -matrix.

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### References

- [ 1 ] M. S. Birman: On the spectrum of singular boundary problem. *Math. Sbornik*, **55**, 124-174 (1961).
- [ 2 ] L. D. Faddeev: Mathematical problems of quantum theory of scattering for three-body system. *Tr. Math. Inst. im. B. A. Cteklova Akad. Nauk. SSSR*, **49** (1963).
- [ 3 ] K. O. Friedrichs: On the perturbation of continuous spectra. *Comm. Pure Appl. Math.*, **1**, 361-406 (1948).
- [ 4 ] T. Kato: Growth properties of solutions of the reduced wave equations with variable coefficient. *Comm. Pure Appl. Math.*, **12**, 402-425 (1959).