151. *Stability in Topological Dynamics*

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1. **Introduction.** In [1], I. Bendixson established many fundamental theorems concerning the flow in the plane defined by a system of differential equations, one of which gives a criterion for the both sense stability of an isolated singular point p: the point p is stable in both senses, positive and negative, if and only if p is a center (in the sense of Bendixson). It has been proved that this criterion is valid if we replace the point p by a simply connected compact invariant set M, isolated from singular points [2]. Since a one-dimensional compact invariant set disjoint from singular points is always a closed orbit, the criterion is transformed as follows: a simply connected compact invariant set M isolated from singular points is stable in both senses, if and only if, in every neighborhood U of M, there exists a non-dense closed invariant set which separates M and the complement of U.

In an earlier paper [3], Ura proved that this transformed condition is also necessary and sufficient for the both sense stability of a compact invariant set of an abstract continuous flow whose phase space is an arbitrary topological space, under only one condition that the space is locally compact, and showed also that the assumptions of M to be isolated from singular points and to be simply connected can also be omitted.

The first object of the present note is to extend again the criterion to more general systems, i.e., to continuous transformation groups whose phase spaces and groups are arbitrary. We shall show that this is done under very slight restrictions on the phase spaces and the phase groups.

As we shall see, the notions related to stability-additivity defined by Ura [3], are also introduced in a very natural way to general transformation groups as to continuous flows; and the second object of this note is to see that general transformation groups have stability-additivity under some slight conditions on the phase spaces and the phase groups.

Remark: All terminologies concerning topology are referred to Bourbaki [4].

2. **Basic notations.** A continuous transformation group is a triple (X, T, π), where X is a topological space, T is a topological

group and π is a continuous mapping of the product space $X\times T$ to the space X, satisfying the following conditions:

(1) $(p, e)\pi=p$, where $p\in X$ and e is the unit element of T.

(2) $((p, t_1)\pi, t_2)\pi=(p, t_1t_2)\pi$, where $p\in X$ and $t_1, t_2\in T$.

Here $(p, t)\pi$ denotes the image by the mapping π of the element (p, t) of $X\times T$ (see [5]).

For simplicity, we shall write $(p, t)\pi=pt$, $pT=\{pt; t\in T\}$, $At=\{pt; p\in A\}$ and $AT=\bigcup_{t\in T} At$, where A is a subset of X. The family of all neighborhoods of a subset M of X will be denoted by $\mathfrak{B}(M)$.

Definition 1. A non-empty subset M of X is said to be *invariant*, if $MT\subset M$.

In accordance with stability in both senses in case of a continuous flow, we shall give the following definition of stability for (X, T, π).

Definition 2. Let M be an invariant subset of X. M is said to be *stable* if, for every neighborhood U of M, there exists a neighborhood V of M such that $VT\subset U$.

3. Stability conditions. As stated in Introduction, our first object is to obtain the stability conditions. To do this, we shall begin with establishing a necessary condition.

Theorem 1. *Suppose that a non-open invariant subset M of X has a neighborhood whose closure is not identical with the whole space X. If M is stable, then for every neighborhood U of M there exists a subset K of X disjoint from M and satisfying the following conditions:*

(1) *K is closed.*

(2) *K separates M and the complement of U, i.e., there exist disjoint, non-empty sets G_1, G_2 both open in $X-K$ such that $X-K=G_1\cup G_2$, $U\supset G_1\supset M$.*

(3) *K is invariant, or empty.*

(4) *K is non-dense.*

(5) *G_1 and G_2 are invariant.*

(6) $K=\mathrm{Fr}.\,G_1$. ($\mathrm{Fr}.\,G_1$ denotes the boundary set of G_1).

Moreover, if X is locally compact and M is compact, then we can impose on K an additional condition:

(7) *K is compact.*

Remark: The topological condition imposed on M stated in the beginning of this theorem is satisfied, if X is a Hausdorff space and M is compact and not open.

Proof. Without loss of generality, we can suppose that $\overline{U}\neq X$. M being stable, there exists an open neighborhood V of M such that $VT\subset U$. Put $K=\mathrm{Fr}.VT$, $G_1=VT$ and $G_2=X-\overline{G}_1$. Then (1), (2), (4), and (6) are obviously satisfied. G_1 is invariant by definition, and so is $\overline{G}_1$. Hence $G_2=X-\overline{G}_1$ is invariant, and if $K=\overline{G}_1-G_1$ is not empty, K is invariant.

Moreover, if X is locally compact and M is compact, we can suppose U is compact. Consequently $K=\mathrm{Fr}.VT$ $(\subset U)$ is compact.

Q.E.D.

The converse of Theorem 1 is trivial, and indeed, the existence of an invariant set G_1 (see (5)) satisfying $U\supset G_1\supset M$ (see (2)) is not other than the definition of the stability of M.

We shall prove that the existence of K satisfying only (1), (2), and (3) is sufficient for the stability of M if the phase group T is connected; more precisely:

Theorem 2. *Suppose that T is connected, and M is an invariant subset of X. Then M is stable if, for every neighborhood U of M, there exists a subset K of X satisfying the following conditions:*

(1) *K is closed.*

(2) *K separates M and the complement of U, i.e., there exist disjoint, non-empty sets G_1, G_2 both open in $X-K$ such that $X-K=G_1\cup G_2$, $U\supset G_1\supset M$.*

(3) *K is invariant, or empty.*

Proof. Suppose $G_1T\not\subset U$, then there exist a point p_0 of G_1 and an element t_0 of T such that $q_0=p_0t_0\notin U$. The point q_0 is situated outside G_1 and inside G_2. Since $p_0T=(p_0T\cap G_1)\cup(p_0T\cap G_2)\cup(p_0T\cap K)$, $p_0T\cap G_1\ni p_0$, $p_0T\cap G_2\ni q_0$, and p_0T is connected, we have $p_0T\cap K\neq\phi$. This is a contradiction to the hypothesis (3). Hence $G_1T\subset U$. Since G_1 is a neighborhood of M, this establishes the stability of M. Q.E.D.

4. Stability-additivity. To consider stability-additivity, we shall explain some notions relating to relative stability; all of them are generalizations of the same notions given in [3].

Definition 3. Let N be any subset of X and M be an invariant subset of X. M is said to be *stable in N* if, for every neighborhood U of M, there exists a neighborhood V of M such that $(V\cap N)T\subset U$.

Definition 4. Let M be an invariant set. M is said to be *componentwise-stable*, if M is stable in every connected component of the complement of M.

The following assertion is easily deduced from the definition.

Theorem 3. *If M is stable, then M is componentwise-stable.*

If the complement of M has only finite number of components, the converse of the theorem is obviously true. We shall show that, even though there are infinitely many components of the complement of M, under some conditions the converse is also true, i.e., the continuous transformation group under consideration has stability-additivity for M:

Theorem 4. *Suppose that X is locally compact and locally con-*

nected, and that T is connected. A compact invariant set M is stable if it is componentwise-stable.

Proof. If M is unstable, we can find a relatively compact open neighborhood U_0 of M such that $UT \not\subset U_0$ for every neighborhood U of M.

We have $UT \cap \mathrm{Fr.}\, U_0 \neq \phi$ for every U of $\mathfrak{B}(M)$, because T is connected. $\mathrm{Fr.}\, U_0$ $(\subset \overline{U}_0)$ being compact, the closure of the filter base $\mathfrak{B} = \{UT \cap \mathrm{Fr.}\, U_0;\ U \in \mathfrak{B}(M)\}$ is not empty. Let p_0 be one of its points, C_0 the connected component of $X - M$ containing the point p_0, and U_1 a neighborhood of M whose closure is contained in U_0. Since X is locally connected, there exists a connected neighborhood V of p_0 disjoint from U_1. V is contained in C_0, and $V \cap UT \neq \phi$ for every U of $\mathfrak{B}(M)$; therefore, there exists a point q of U such that $qT \cap V \neq \phi$. Since qT is connected, we have $qT \subset C_0$, whence $q \in C_0$. This means $(U \cap C_0)T \not\subset U_1$, $U \in \mathfrak{B}(M)$, which shows that M is not componentwise-stable. Q.E.D.

References

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