175. On the Absolute Nörlund Summability of a Fourier Series*)

By H. P. DIKSHIT

University of Allahabad, India

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1.1. *Definitions.* Let $\sum a_n$ be a given infinite series and $\{s_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of constants, real or complex and let us write

$$P_n=p_0+p_1+\cdots+p_n;\quad P_{-1}=p_{-1}=0.$$

The sequence to sequence transformation:

$$t_n=\sum_{\nu=0}^{n} p_{n-\nu}s_\nu/P_n;\quad P_n\neq 0, \tag{1.1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means[1] of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if $\{t_n\}\in BV$, that is, $\sum_n |t_n-t_{n-1}|\leqq K$.[2]

1.2. Let $f(t)$ be a periodic function with period 2π, and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume, without any loss of generality, that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt=0 \tag{1.2.1}$$

and

$$f(t)\sim\sum_n (a_n\cos nt+b_n\sin nt)=\sum_n A_n(t). \tag{1.2.2}$$

We write throughout:

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\};$$

$$c_{n,k}=\{\sin(n-k)t\}/(n-k);$$

$$R_n=(n+1)p_n/P_n;$$

$$T_n=1/R_n=P_n(n+1)^{-1}/p_n;$$

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1) Nörlund [3].

2) Mears [1].

$S_n=\sum_{\nu=0}^{n} P_\nu(\nu+1)^{-1}/P_n;$

$\Delta\sigma_n=\sigma_{n+1}-\sigma_n$, for any sequence σ_n.

$\tau=[1/t]$, that is, the greatest integer contained in $1/t$.

K denotes a positive constant not necessarily the same at each occurrence.

1.3. *Introduction.* Pati[3] has proved the following theorem concerning the summability $|N, p_n|$ of the Fourier series of $f(t)$, at $t=x$.

Theorem A. If $\phi(t)\in BV(0, \pi)$[4] and $\{p_n\}$ is a positive monotonic sequence such that $P_n\to\infty$ as $n\to\infty$, and $\{R_n\}\in BV$ and $\{S_n\}\in BV$, then the Fourier series of $f(t)$, at $t=x$, is summable $|N, p_n|$.

We observe that in the case in which $\{p_n\}$ is positive monotonic non-decreasing, $\{R_n\}\in BV$ implies $\{T_n\}\in BV$, and $\{T_n\}\in BV$ in its turn implies $\{S_n\}\in BV$. This follows when we observe that

$$S_n=\sum_{\nu=0}^{n} T_\nu p_\nu/P_n,$$

and appeal to the result (Mohanty [2], Lemma 4):

If $\mu_n>0$, $\lambda_n=\mu_1+\mu_2+\cdots+\mu_n$, and

$$d_n=\{\mu_1 c_1+\cdots+\mu_n c_n\}/\lambda_n,$$

then, $\{d_n\}\in BV$ whenever $\{c_n\}\in BV$. Hence Pati actually uses the hypothesis that $\{R_n\}\in BV$, in the case: $\{p_n\}$ is monotonic non-decreasing. However in this case, $\{T_n\}\in B$.[5] And we therefore have $\{T_n\}\in BV$. Further if we assume that $\{R_n\}\in B$, and $\{T_n\}\in BV$, then we indeed get $\{R_n\}\in BV$. Thus in the present case the set of hypotheses used by Pati, viz. $\{R_n\}\in BV$ and $\{S_n\}\in BV$, are equivalent to the hypotheses $\{R_n\}\in B$ and $\{T_n\}\in BV$.

The object of the present paper is to provide an appreciably brief proof of Theorem A, in the case in which $\{p_n\}$ is a positive monotonic non-decreasing sequence, in the following equivalent form.

THEOREM. *If $\phi(t)\in BV(0, \pi)$ and $\{p_n\}$ is a positive, monotonic non-decreasing sequence such that $\{T_n\}\in BV$ and $\{R_n\}\in B$, then the Fourier series of $f(t)$, at $t=x$, is summable $|N, p_n|$.*

1.3. We shall require the following lemmas for the proof of the Theorem.

LEMMA 1.[6] *If $\lambda_{n,k}(t)$ be any function of n, k, and t, then*

$$\sum_{k=0}^{n-1}\left(\frac{P_n}{n+1}p_k-p_n\frac{P_k}{k+1}\right)\lambda_{n,k}(t)=p_n\sum_{\nu=0}^{n-1}\Delta T_\nu\sum_{k=0}^{\nu}p_k\lambda_{n,k}(t).$$

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3) Pati [5].

4) By '$F(t)\in BV(h, k)$' we mean that $F(t)$ is a function of bounded variation over the interval (h, k).

5) That is, $\{T_n\}$ is a bounded sequence. We follow such symbolism consistently.

6) This lemma and its proof were given by Professor L. S. Bosanquet in a communication to Dr. Pati.

Proof. We write

$$\sum_{k=0}^{n-1}\left(\frac{P_n}{n+1}p_k-p_n\frac{P_k}{k+1}\right)\lambda_{n,k}(t)=p_n\sum_{k=0}^{n-1}p_k(T_n-T_k)\lambda_{n,k}(t)$$
$$=p_n\sum_{k=0}^{n-1}p_k\lambda_{n,k}(t)\sum_{\nu=k}^{n-1}\Delta T_\nu$$
$$=p_n\sum_{\nu=0}^{n-1}\Delta T_\nu\sum_{k=0}^{\nu}p_k\lambda_{n,k}(t).$$

LEMMA 2.[7] *Uniformly for* $0<t\leqq\pi$,

$$\left|\sum_{m}^{n}\sin\nu t/\nu\right|\leqq K,$$

where m *and* n *are positive integers such that* $m\leqq n$.

LEMMA 3.[8] *If* $\{p_n\}$ *is a positive monotonic non-decreasing sequence such that* $\{R_n\}\in B$, *then uniformly for* $0<t\leqq\pi$,

$$\sum_n\frac{p_n}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}\frac{P_k}{k+1}\sin(n-k)t\right|\leqq K.$$

LEMMA 4. *For positive* $\{p_n\}$,

$$\sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\leqq\frac{1}{P_\nu}.$$

This is evident since $p_n=P_n-P_{n-1}$ and P_n is monotonic increasing.

1.4. *Proof of the theorem.* As in Pati [4], in order to prove our theorem we have to show that, uniformly for $0<t\leqq\pi$,

$$\sum=\sum_n\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)c_{n,k}\right|\leqq K.$$

We have

$$\sum\leqq\sum_n\frac{(n+1)}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}\left(\frac{P_n}{n+1}p_k-p_n\frac{P_k}{k+1}\right)c_{n,k}\right|$$
$$+\sum_n\frac{(n+1)}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}\left(\frac{P_k}{k+1}p_n-p_n\frac{P_k}{n+1}\right)c_{n,k}\right|$$
$$=\sum\nolimits_1+\sum\nolimits_2,\quad\text{say.}$$

Now,

$$\sum\nolimits_2=\sum_n\frac{p_n}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}\frac{P_k}{k+1}\sin(n-k)t\right|\leqq K,$$

by Lemma 3.

Also,

7) Titchmarsh [6], §1.76.
8) This result is due to Pati. See proof of $\sum_2\leqq K$, in Pati [5].

$$\begin{aligned}\sum_1 &= \sum_n \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left(\frac{P_n}{n+1} p_k - p_n \frac{P_k}{k+1} \right) \sin (n-k)t \left(1 + \frac{k+1}{n-k}\right) \right| \\ &\leqq \sum_n \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left(\frac{P_n}{n+1} p_k - p_n \frac{P_k}{k+1} \right) \sin (n-k)t \right| \\ &\quad + \sum_n \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left(\frac{P_n}{n+1} p_k - p_n \frac{P_k}{k+1} \right) (k+1) c_{n,k} \right| \\ &= \sum_{11} + \sum_{12}, \quad \text{say.}\end{aligned}$$

By Lemma 1,

$$\begin{aligned}\sum_{11} &= \sum_n \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \Delta T_\nu \sum_{k=0}^{\nu} p_k \sin (n-k)t \right| \\ &\leqq K \sum_n \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} |\Delta T_\nu| \sum_{k=0}^{\nu} p_k \\ &= K \sum_n \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} |\Delta T_\nu| P_\nu \\ &= K \sum_\nu |\Delta T_\nu| P_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \\ &\leqq K \sum_\nu |\Delta T_\nu|, \quad \text{by Lemma 4,} \\ &\leqq K,\end{aligned}$$

by the hypothesis that $\{T_n\} \in BV$.

Also by Lemma 1,

$$\begin{aligned}\sum_{12} &= \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \Delta T_\nu \sum_{k=0}^{\nu} (k+1) p_k c_{n,k} \right| \\ &\leqq K \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} |\Delta T_\nu| (\nu+1) p_\nu,\end{aligned}$$

by Abel's Lemma and Lemma 2, since $p_n(n+1)$ is monotonic non-decreasing. And therefore

$$\begin{aligned}\sum_{12} &\leqq K \sum_\nu |\Delta T_\nu| (\nu+1) p_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \\ &\leqq K \sum_\nu |\Delta T_\nu| R_\nu, \quad \text{by Lemma 4,} \\ &\leqq K \sum_\nu |\Delta T_\nu| \\ &\leqq K,\end{aligned}$$

by the hypotheses that $\{R_n\} \in B$ and $\{T_n\} \in BV$.

This completes the proof of the theorem.

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