174. A Tauberian Theorem for (J, p_n) Summability*)

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§1. We suppose throughout that

$$p_n \geq 0, \qquad \sum_{n=0}^{\infty} p_n = \infty,$$

and that the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$

is *1*. Given any series

(1) $$\sum_{n=0}^{\infty} a_n,$$

with the sequence of partial sums $\{s_n\}$, we shall use the notation:

(2) $$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n.$$

If the series (2) is convergent in the open interval (0, 1), and if

$$\lim_{x \to 1-0} \frac{p_s(x)}{p(x)} = s,$$

we say that the series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is smmable (J, p_n) to s. As is well known, this method of summability is regular. (See, Borwein [1], Hardy [2], p. 80.)

Now we write

$$P_n = p_0 + p_1 + \cdots + p_n, \quad n = 0, 1, \cdots,$$

and

(3) $$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu, \quad n = 0, 1, \cdots,$$

with $p_n > 0$. If $\{t_n\}$ is convergent to s, we say that the series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is summable $(\bar{N}, p_n)$ to s. This method of summability is also regular, and is equivalent to the Riesz method $(R, P_{n-1}, 1)$. (See, Hardy [2], pp. 57, 86, Jurkat [4], Kuttner [5,6].)

We shall first state the following

Theorem 1. *$(\bar{N}, p_n)$ implies*[1] *(J, p_n).*

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*) Dedicated to Professor Kinjirô Kunugi for his 60th Birthday.

1) Given two summability methods A, B, we say that A *implies* B if any series or sequence summable A is summable B to the same sum. We say that A is *equivalent* to B if A implies B and B implies A.

The proof of this theorem may be deduced from a general theorem, however we shall give here a sketch of a brief proof. (See, e.g., Hobson [3], p. 181.)

From (3) we get

$$t_nP_n - t_{n-1}P_{n-1} = p_ns_n, \quad n=0, 1, \cdots,$$

with $t_{-1}=P_{-1}=0$. Hence

$$\begin{aligned} p_s(x) &= \sum_{n=0}^{\infty} p_ns_nx^n \\ &= \sum_{n=0}^{\infty} (t_nP_n - t_{n-1}P_{n-1})x^n \\ &= (1-x) \sum_{n=0}^{\infty} t_nP_nx^n \end{aligned}$$

from the assumption of the theorem. Now since

$$\begin{aligned} \frac{p_s(x)}{p(x)} &= \frac{(1-x)\sum_{n=0}^{\infty} t_nP_nx^n}{\sum_{n=0}^{\infty} p_nx^n} \\ &= \frac{\sum_{n=0}^{\infty} t_nP_nx^n}{\sum_{n=0}^{\infty} P_nx^n} = \frac{P_t(x)}{P(x)}, \end{aligned}$$

we have, again from the assumption of the theorem,

$$\lim_{x\to 1-0} \frac{p_s(x)}{p(x)} = \lim_{x\to 1-0} \frac{P_t(x)}{P(x)} = s,$$

which proves the theorem.

§ 2. Concerning the $(\bar{N}, p_n)$ summability we know the following Tauberian

Theorem 2. *Suppose that*

$$p_n > 0, \quad n=0, 1, \cdots,$$

that

$$a_n = O\left(\frac{p_n}{P_n}\right),$$

and that the series (1) *is summable* $(R, P_n, 1)$. *Then* (1) *converges to the same sum.* (See, Hardy [2], p.124.)

Since $(\bar{N}, p_n)$ implies (J, p_n), we can expect Tauberian theorems of the similar type for the (J, p_n) summability. We shall prove here the following

Theorem 3. *Suppose that*

$$\frac{\sum_{n=0}^{m} p_n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} = O(1) \qquad \textit{for} \quad m\to\infty, \tag{4}$$

that

$$(5) \qquad 0<p_n \leq M, \quad n=0, 1, \cdots,$$

with some constant M, *and that*

$$(6) \qquad \frac{n}{P_n}=O(1).$$

Suppose that the series (1) *is summable* (J, p_n) *to* s, *and that*

$$(7) \qquad a_n=o\left(\frac{p_n}{P_n}\right).$$

Then (1) *converges to* s.

Proof. We have, for $0<x<1$,

$$s_m-\frac{p_s(x)}{p(x)}=\frac{\sum_{n=0}^{\infty} s_m p_n x^n}{\sum_{n=0}^{\infty} p_n x^n}-\frac{\sum_{n=0}^{\infty} s_n p_n x^n}{\sum_{n=0}^{\infty} p_n x^n}$$

$$=\frac{\sum_{n=0}^{\infty} (s_m-s_n) p_n x^n}{\sum_{n=0}^{\infty} p_n x^n}$$

$$=\frac{\sum_{n=0}^{m-1} (s_m-s_n) p_n x^n}{\sum_{n=0}^{\infty} p_n x^n}+\frac{\sum_{n=m+1}^{\infty} (s_m-s_n) p_n x^n}{\sum_{n=0}^{\infty} p_n x^n}$$

$$=I+J, \quad \text{say.}$$

Here we get

$$|I| \leq \frac{\sum_{n=0}^{m-1} |s_m-s_n| p_n}{\sum_{n=0}^{\infty} p_n x^n}$$

$$\leq \frac{1}{\sum_{n=0}^{\infty} p_n x^n}\left\{p_1 \frac{|a_1| p_0}{p_1}+p_2 \frac{|a_2|(p_0+p_1)}{p_2}+\cdots+\right.$$

$$\left.+p_m \frac{|a_m|(p_0+p_1+\cdots+p_{m-1})}{p_m}\right\},$$

and therefore, when $x=1-\frac{1}{m}$,

$$|I| \leq \frac{P_m}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} \cdot \frac{1}{P_m} \cdot\left\{p_1 \frac{|a_1| P_0}{p_1}+\right.$$

$$\left.+p_2 \frac{|a_2| P_1}{p_2}+\cdots+p_m \frac{|a_m| P_{m-1}}{p_m}\right\}.$$

Now, from (7), we see

$$\frac{|a_m| P_{m-1}}{p_m}=o(1) \quad \text{for} \quad m\to\infty.$$

Hence, according to (4), we have

(8) $$I=o(1) \quad \text{for} \quad m\to\infty.$$

Next we shall estimate J. For any ε, $\varepsilon>0$, let m be so chosen that

$$|a_n| \leq \varepsilon \frac{p_n}{P_n}$$

for $n>m$, then

$$\begin{aligned}|s_m-s_n| &\leq \varepsilon\left\{\frac{p_{m+1}}{P_{m+1}}+\frac{p_{m+2}}{P_{m+2}}+\cdots+\frac{p_n}{P_n}\right\}\\ &=\varepsilon Q_n, \quad \text{say.}\end{aligned}$$

Therefore we have

(9) $$\begin{aligned}|J| &\leq \frac{\varepsilon \sum_{n=m+1}^{\infty} Q_n p_n x^n}{\sum_{n=0}^{\infty} p_n x^n}\\ &=\frac{\varepsilon \sum_{n=m+1}^{\infty} Q_n p_n \left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n},\end{aligned}$$

if x be chosen to be equal to $1-\frac{1}{m}$. Since

$$Q_n \leq \frac{P_n-P_m}{P_m}=\frac{P_n}{P_m}-1,$$

we get

$$\begin{aligned}|J| &\leq \frac{\varepsilon \frac{1}{P_m} \sum_{n=m+1}^{\infty} P_n p_n \left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n}\\ &=\frac{\varepsilon P_m \frac{1}{P_m^2} \sum_{n=m+1}^{\infty} P_n p_n \left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n}\\ &\leq \varepsilon M \frac{1}{P_m^2} \sum_{n=m+1}^{\infty} P_n \left(1-\frac{1}{m}\right)^n \text{ [2]}\end{aligned}$$

from (4) and (5). Also, again using (5), we have

2) We use M to denote a constant, possibly different at each occurrence.

$$
\begin{aligned}
(10)\qquad |J| &\leq \varepsilon M \frac{1}{P_m^2} \sum_{n=m+1}^{\infty} n\left(1-\frac{1}{m}\right)^n \\
&\leq \varepsilon M \frac{1}{P_m^2} \int_m^\infty x\left(1-\frac{1}{m}\right)^x dx \\
&\leq \varepsilon M \frac{m^2}{P_m^2} \\
&\leq \varepsilon M,
\end{aligned}
$$

for large m, from (6).

Letting m increase indefinitely, we have

$$\lim_{m\to\infty} s_m = \lim_{x\to 1-0} \frac{p_s(x)}{p(x)} = s$$

from (8) and (10), which proves the theorem.

§ 3. The assumptions of Theorem 3 seem to be very complicated, however it follows from this theorem the following

Corollary. *Suppose that there exist two numbers σ, M such that*

$$(11)\qquad 0<\sigma\leq p_n\leq M\,, \quad n=0, 1, \cdots .$$

Suppose that the series (1) *is summable* (J, p_n) *to* s, *and that*

$$(7)\qquad a_n = o\left(\frac{p_n}{P_n}\right).$$

Then (1) *converges to* s.

Proof. It suffices to prove that (11) implies (4) and (6). From (11), we see

$$
\begin{aligned}
\frac{\sum_{n=0}^{m} p_n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} &\leq M \frac{m+1}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} \\
&\leq M \frac{\sum_{n=0}^{\infty}\left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} \\
&\leq \frac{M}{\sigma} < \infty
\end{aligned}
$$

for large m. Finally we see, from (11),

$$\frac{n}{P_n} \leq \frac{n}{(n+1)\sigma} < \frac{1}{\sigma}$$

for large n. We reach the desired conclusion.

Remark. In the corollary, the condition (7) may be replaced by

$a_n = o\left(\frac{1}{n}\right).$

References

[1] D. Borwein: On methods of summability based on power series. Proc. Roy. Soc. Edinburgh, Sect. A, **64**, 342-349 (1957).

[2] G. H. Hardy: Divergent Series. Oxford (1949).

[3] E. W. Hobson: The Theory of Functions of a Real Variable and the Theory of Fourier Series. vol. 2. Cambridge (1926).

[4] W. Jurkat: Über Rieszsche Mittel mit unstetigem Parameter. Math. Z., **55**, 8-12 (1952).

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[6] ——: A Tauberian theorem for discontinuous Riesz means (II). Ibid., **39**, 643-648 (1964).