172. *Semigroups Whose Regular Representation is a Group*[1)]

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The set of all mappings ρ_x, defined by $a\rho_x = ax$, is called the regular representation of S. The purpose of this note is to determine all semigroups whose regular representation is a group.

A left group is a semigroup with a right identity and with left solvability. In [1] Clifford proved that any left group is a direct product GxL of a group G and a left zero semigroup L. See [3] for other equivalent definitions.

Lemma 1. *If T is the regular representation of a left group GxL, then $T \simeq G$.*

Lemma 2. *If S is a semigroup and if T is its regular representation, then T is a permutation group if and only if S is a left group.*

Lemma 3. *If T is a permutation group on a set S, then there exists a binary operation on S such that S is a semigroup with T as its regular representation if and only if T satisfies the condition that, for all $\alpha, \beta \in T$ and for all $x \in S$, $x\alpha = x\beta$ implies that $\alpha = \beta$.*

To demonstrate the binary operation in Lemma 3, we let $\{S_i\}$ be the collection of transitivity components of T. We then select from each S_i an element e_i. Now, for each $x \in S_i$ there exists, by assumption, a unique element $\alpha \in T$ such that $e_i\alpha = x$. Denoting this α by $x\varphi_i$, we get a mapping, for each i, from S_i into T. The operation $x \cdot y = x(y\varphi_i)$ if $y \in S_i$, makes S a semigroup with T as its regular representation.

If T is a transformation semigroup on a set S, let S^* be the set of all elements of S which are in the range of some member of T, and let T^* be the set of all elements of T restricted to S^*.

Lemma 4. *If T is a transformation group on a set S, then T^* is a permutation group on S^*.*

Theorem 1. *If T is a transformation group on a set S, then there exists a binary operation on S such that S is a semigroup with T as its regular representation if and only if T satisfies the*

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1) This paper was presented by the author at the 1964 Summer Meeting of the American Mathematical Society at Amherst. The detailed proof will appear elsewhere.

condition that, for all α, $\beta \in T$ and for all $x \in S$, $x\alpha = x\beta$ implies $\alpha = \beta$.

Definition. Let S be a semigroup. For each $x \in S$, let A_x be a set containing x such that the sets A_x, $x \in S$, are pairwise disjoint. Let $S' = \cup \{A_x : x \in S\}$, and define $a \cdot b = xy$ where $a \in A_x$ and $b \in A_y$. The semigroup $(S', \cdot)$ is called an inflation of S.

Theorem 2. *If S is a semigroup and if T is its regular representation, then T is a group if and only if S is an inflation of a left group.*

In the proofs of Theorems 1 and 2 we use Lemma 4 so that we may apply Lemmas 2 and 3 respectively.

Corollary. *If S is a semigroup such that both its regular representation and its anti-representation* [2, p.9] *are groups, then S is an inflation of a group.*

Theorem 3. *If S is a semigroup such that its regular representation T is a group, then T is the maximal group homomorphic image of S.*

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Addendum: Theorem 2 also can be proved by using the concept "M-inversive" due to Yamada (see [Kodai Math. Sem. Rep., 7(1955) 49–52] or [1, p.98]). Furthermore the author has obtained that S is M-inversive if and only if T is a right group.

References

[1] A. H. Clifford: A system arising from a weakened set of group postulates. Annals of Math., **34**, 865–871 (1933).

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[3] T. Tamura, R. B. Merkel, and J. F. Latimer: The direct product of right singular semigroups and certain groupoids. Proc. Amer. Math. Soc., **14**, 118–123 (1963).