171. *On a Theorem of Brauer*

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The purpose of this paper is to give a simple proof of a theorem of Brauer concerning the principal blocks of characters of finite groups ([4], Theorem 3, see also [3]).

We refer to Brauer [1], [2]; Brauer-Nesbitt [6]; Osima [8], and Curtis-Reiner [7] as for basic concepts and theorems about the blocks of characters of finite groups.

1. Let G be a group of a finite order and let p be a fixed prime number. We choose the algebraic number field Ω such that the absolutely irreducible representations of G can be written with coefficients in Ω. Let $\mathfrak{p}$ be a prime ideal divisor of p in Ω and let $\mathfrak{o}_\mathfrak{p}$ be the ring of all $\mathfrak{p}$-integers of Ω, and $\bar{\Omega}$ the residue class field of $\mathfrak{o}_\mathfrak{p}$ (mod $\mathfrak{p}$). The residue class map of $\mathfrak{o}_\mathfrak{p}$ onto $\bar{\Omega}$ will be denoted by an asterisk; $\alpha \to \alpha^*$.

If M is a subset of G, we write $|M|$ for the number of elements of M. The centralizer of M in G will be denoted by $C_G(M)$ and the normalizer of M in G by $N_G(M)$.

The group algebra of G over $\bar{\Omega}$ will be denoted by $\Gamma(G)$ and its center by $Z(G)$. If M is a subset of G, we write $[M]$ for the element of $\Gamma(G)$ defined by

$$[M]=\sum_{m\in M} m. \tag{1.1}$$

If $K_1, K_2, \cdots, K_m$ are the conjugate classes of G, the elements $[K_1], [K_2], \cdots, [K_m]$ form a basis of $Z(G)$. Let us denote by $\psi_0, \psi_1, \cdots, \psi_{s-1}$ the distinct linear characters of $Z(G)$. The m (absolutely) irreducible characters $\chi_0=1, \chi_1, \cdots, \chi_{m-1}$ of G are distributed into s blocks $B_0, B_1, \cdots, B_{s-1}$ for p. There exists a one-to-one correspondence between the set of blocks of G and the set of linear characters of $Z(G)$. The block $B_0=B_0(G)$ of G containing the principal character $\chi_0=1$ is called the principal block of G.

Since each primitive idempotent of $Z(G)$ is associated with a block of G, we shall denote by δ_τ the primitive idempotent associated with B_τ. We then have

$$\psi_\tau(\delta_\sigma)=\begin{cases}1, & \tau=\sigma\\ 0, & \tau\neq\sigma.\end{cases} \tag{1.2}$$

If we set

(1.3) $$\omega_i([K_\alpha]) = |K_\alpha| \chi_i(u_\alpha)/\chi_i(1)$$

where u_α is an element in the class K_α, then the map ω_i^* of $Z(G)$ into $\bar{\Omega}$ defined by $\omega_i^*([K_\alpha]) = (\omega_i([K_\alpha]))^*$ is a linear character of $Z(.G)$ Two characters χ_i and χ_j belong to the same block, if and only if $\omega_i^*([K_\alpha]) = \omega_j^*([K_\alpha])$ for all p-regular classes K_α of G, i.e. for classes of G which consist of elements whose order is prime to p. For $\chi_i \in B_\tau$, we have

(1.4) $$\psi_\tau = \omega_i^*.$$

Let V be a set of p-regular elements of G. We have $|V| \not\equiv 0 \pmod{p}$ ([5], [6]). Hence if we set

(1.5) $$\varepsilon_0 = (1/|V|^*)[V],$$

then $\varepsilon_0 \in Z(G)$ and we have the following

Lemma 1. $\varepsilon_0 - \delta_0 \in \operatorname{rad} Z(G)$ *where* $\operatorname{rad} Z(G)$ *denotes the radical of* $Z(G)$.

Proof. We have $\chi_0([V]) = |V|$, and $\chi_i([V]) = 0$ for $\chi_i \notin B_0(G)$ ([9], Theorem 2). Hence $\omega_0^*(\varepsilon_0) = 1$, and $\omega_i^*(\varepsilon_0) = 0$ for $\chi_i \notin B_0(G)$. It follows from (1.2) and (1.4) that $\psi_\tau(\varepsilon_0 - \delta_0) = 0$ for every ψ_τ. This implies that $\varepsilon_0 - \delta_0 \in \operatorname{rad} Z(G)$.

If p is prime to the order $|G|$, we have $\varepsilon_0 = \delta_0$. The generalization of Lemma 1 for any block of G and its applications will be shown in another paper.

Let Q be a p-subgroup of G. We shall say that $u, v \in G$ are Q-conjugate, if there exists $\pi \in Q$ such that $v = \pi^{-1} u \pi$. Let $L_1, L_2, \cdots, L_t$ be the Q-conjugacy classes of G. The $|L_i|$ is a power of p and $|L_i| = 1$, if and only if L_i consists of an element in $C_G(Q)$. Hence $|V \cap C_G(Q)|^* = |V|^*$. Now assume that Q is normal in G. If $K_\alpha \cap C_G(G) = \phi$ for $K_\alpha \subseteqq V$, then $[K_\alpha] \in \operatorname{rad} Z(G)$ ([8], p. 183). Hence if we set

(1.6) $$\eta_0 = (1/|U|^*)[U]$$

where $U = V \cap C_G(Q)$, then we obtain readily

Lemma 2. $\eta_0 - \delta_0 \in \operatorname{rad} Z(G)$.

In particular, if G contains a normal p-Sylow subgroup Q, then we can prove that $\eta_0 = \delta_0$. This will be shown also in another paper.

2. Let H be a subgroup of G and let h be a linear function in $Z(H)$. Then, as in Brauer [2] we define a linear function h^G in $Z(G)$ by

(2.1) $$h^G([K_\alpha]) = h([K_\alpha \cap Z(H)]).$$

If K is a subgroup of G such that $H \subseteqq K$, then we have $(h^K)^G = h^G$. Denote by ψ' the linear character of $Z(H)$ associated with the block b, and if $\psi = (\psi')^G$ is a linear character of $Z(G)$, then we say that b^G is defined and we set $b^G=B$ where B is a block of $Z(G)$ associated with $(\psi')^G$.

In the following, if H and K are subgroups of G, we shall indicate by $H\subseteqq_c K$ that H is contained in some conjugate of K. Let Q be a p-subgroup of G and let H be a subgroup of G such that $QC_G(Q)\subseteqq H\subseteqq N_G(Q)$. The map σ: $[K_\alpha]\to[K_\alpha\cap C_G(Q)]$ defines a homomorphism of $Z(G)$ into $Z(H)$ ([1], 7B). As an application, we obtain the following Lemma ([2], 2A).

Lemma 3. *Let Q be a p-subgroup of G and let H be a subgroup of G such that $QC_G(Q)\subseteqq H\subseteqq N_G(Q)$. Let b be any block of H. Then b^G is defined. If B is a block of G with the defect group D such that $Q\subseteqq_c D$, then there exist blocks b of H for which $b^G=B$.*

Now we shall prove the following

Lemma 4. *Let H have the same significance as in Lemma 3. Let b be a block of H. Then $b^G=B_0(G)$, if and only if $b=B_0(H)$.*

Proof. Denote by δ_0' the idempotent of $Z(H)$ associated with $B_0(H)$. To prove Lemma 4, we need show only that $\sigma(\delta_0)=\delta_0'$. From our assumption we see that Q is normal in H and that $C_H(Q)=C_G(Q)$. If we set $V\cap H=V'$ and $V'\cap C_G(Q)=U'$, then it follows from Lemma 2 that $\eta_0'-\delta_0'\in \operatorname{rad} Z(H)$ where $\eta_0'=(1/|U'|^*)[U']$. Since $U'=V\cap C_G(Q)$, we have

$$\sigma(\varepsilon_0)=(1/|V|^*)[V\cap C_G(Q)]=(1/|U'|^*)[U']=\eta_0'.$$

It follows from Lemma 1 that $\eta_0'-\sigma(\delta_0)\in\operatorname{rad} Z(H)$ and hence $\sigma(\delta_0)-\delta_0'\in\operatorname{rad} Z(H)$. Since $\sigma(\delta_0)$ is an idempotent of $Z(H)$, this implies that $\sigma(\delta_0)=\delta_0'$.

Let π be a fixed p-element of G. If v is a p-regular element of $C_G(\pi)$, we have

$$\chi_i(\pi v)=\sum_\rho d_{i\rho}^\pi\varphi_\rho^\pi(v) \tag{2.2}$$

for $\chi_i\in B$. Here φ_ρ^π ranges over the modular irreducible characters of the blocks b of $C_G(\pi)$ for which $b^G=B$. The $d_{i\rho}^\pi$ are called the generalized decomposition numbers of G. We obtain the following theorem ([4], Corollary 4).

Theorem 1. *If $B=B_0(G)$, then φ_ρ^π in* (2.2) *ranges over the modular irreducible characters of $B_0(C_G(\pi))$.*

Proof. Apply Lemma 4 to G and its subgroup $H=C_G(\pi)$.

Let Q be a p-subgroup of G and let H be a subgroup of G such that $QC_G(Q)\subseteqq H$. Then we have

Lemma 5. *Let b be a block of H with the defect group D (in H). If $Q\subseteqq_c D$, then $b^G=B$ is defined.*

Proof. If we apply Lemma 3 to H and its subgroup $\tilde{G}=QC_G(Q)=QC_H(Q)$, we see that there exist blocks $\tilde{b}$ of $\tilde{G}$ for which $\tilde{b}^H=b$. Again, applying Lemma 3 to G and its subgroup $\tilde{G}$, $\tilde{b}^G=B$ is defined.

Hence

$$b^G=(\tilde{b}^H)^G=\tilde{b}^G=B.$$

Theorem 2. *Let Q be a p-subgroup of G and let H be a* sub-*group of G such that $QC_G(Q)\subseteqq H$. Let b be a block of H with the defect group D. If $Q\subseteqq_c D$, then $b^G=B_0(G)$, if and only if $b=B_0(H)$.*

Proof. Assume first that $b=B_0(H)$. Applying Lemma 4 to H and its subgroup $\tilde{G}=QC_G(Q)$, we have $(B_0(\tilde{G}))^H=B_0(H)$. On the other hand, applying Lemma 4 to G and its subgroup $\tilde{G}$, we have $(B_0(\tilde{G}))^G=B_0(G)$. Hence $(B_0(H))^G=(B_0(\tilde{G})^H)^G=(B_0(\tilde{G}))^G=B_0(G)$. Conversely, assume that $b^G=B_0(G)$. There exist by Lemma 3 blocks $\tilde{b}$ of $\tilde{G}$ for which $\tilde{b}^H=b$. Hence $(\tilde{b}^H)^G=\tilde{b}^G=B_0(G)$. It follows from Lemma 4 that $\tilde{b}=B_0(\tilde{G})$ and hence we have $b=(B_0(\tilde{G}))^H=B_0(H)$.

If we set $Q=D$ in Theorem 2, we obtain ([4], Theorem 3).

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