169. *On the Permutability of Congruences on Algebraic Systems*[*)]

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K. Shoda discussed in his papers [8], [10], and his book [9] the structure of an algebraic system $\mathfrak{A}$ under the following conditions:

I. $\mathfrak{A}$ has a zero-element, i. e. $\mathfrak{A}$ has a one-element subsystem.

II. Any subsystem of $\mathfrak{A}$ generated by two normal subsystems of $\mathfrak{A}$ is a normal subsystem of $\mathfrak{A}$.

III. Any natural meromorphism between any two residue class systems of $\mathfrak{A}$ is classable.

G. Birkhoff discussed in his book [1] the structure of an algebraic system $\mathfrak{A}$ under the following conditions:

I. $\mathfrak{A}$ has a one-element subsystem.

III*. Any two congruences on $\mathfrak{A}$ are permutable.

K. Shoda told the author that the conditions III and III* are equivalent as stated in the introduction of the author's paper [2]. The conditions III and III* played the important role in their structure theories of algebraic systems.

A. I. Mal'cev proved in his paper [7] the following

Theorem. *Let A be a set of composition-identities with respect to a system V of compositions. Then the following two conditions are equivalent*:

(a) *Any two congruences on any A-algebraic system are permutable.*

(b) *There exists a derived composition $f(\xi, \eta, \zeta)$ of V such that*

$$f(\xi, \eta, \eta) \overset{A}{=} \xi^{1)} \text{ and } f(\xi, \xi, \eta) \overset{A}{=} \eta.$$

Moreover J. Lambek remarked in his paper [6] that *each of the conditions* (a) *and* (b) *is equivalent to the following condition*:

(c) *Any meromorphism between any two A-algebraic systems is classable.*

A. W. Goldie and the author have discussed in the papers [2], [3], [4], and [5] the structure of algebraic systems. The weak permutability and the local permutability of congruences have played the leading role in the theories of A. W. Goldie and of the author.

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1) $f(\xi, \eta, \eta) \overset{A}{=} \xi$ denotes the fact that $f(x, y, y)=x$ holds for any elements x and y in any A-algebraic system, i.e. $f(\xi, \eta, \eta)=\xi$ is derived from A.

In this note, we shall show the theorems concerning the weak permutability and the local permutability of congruences which are analogous to the above Mal'cev's theorem and analogous to the Lambek's remark. By these theorems, we can understand the fact that the weak permutability and the local permutability are closely connected with the existence of derived compositions with units in some meaning.

Let $\mathfrak{A}$ be an algebraic system, and let θ and φ be two congruences on $\mathfrak{A}$. We now consider the following conditions on relation between θ and φ at an element a in $\mathfrak{A}$:

$K_1(a)$. For each element x in $\mathfrak{A}$, there exists an element y in $\mathfrak{A}$ such that $a \overset{\theta}{\sim} y \overset{\varphi}{\sim} x$ if and only if there exists an element z such that $a \overset{\varphi}{\sim} z \overset{\theta}{\sim} x$.

$K_2(a)$. If $x \overset{\theta}{\sim} a \overset{\varphi}{\sim} y$, then there exists an element z in $\mathfrak{A}$ such that $x \overset{\varphi}{\sim} z \overset{\theta}{\sim} y$.

θ and φ are said to be weakly permutable at a if and only if θ and φ satisfy the condition $K_1(a)$. θ and φ are said to be locally permutable at a if and only if θ and φ satisfy the conditions $K_1(a)$ and $K_2(a)$. θ and φ are said to be permutable if and only if θ and φ satisfy the condition $K_2(a)$ for every element a in $\mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two algebraic systems with respect to the same system of compositions. Let $\mathfrak{M}$ be a subsystem of the direct product $\mathfrak{A}\times\mathfrak{A}'$ such that for any element a in $\mathfrak{A}$ there exists an element a' in $\mathfrak{A}'$ satisfying $(a, a')\in\mathfrak{M}$, and that for any element b' in $\mathfrak{A}'$ there exists an element b in $\mathfrak{A}$ satisfying $(b, b')\in\mathfrak{M}$. The correspondence μ between $\mathfrak{A}$ and $\mathfrak{A}'$ which is defined by such a graph $\mathfrak{M}$ is called a meromorphism between $\mathfrak{A}$ and $\mathfrak{A}'$. We now consider the following conditions on the meromorphism μ at an element (a, a') in $\mathfrak{M}$:

$M_1(a, a')$. For each element (b, b') in $\mathfrak{M}$, (a, b') is contained in $\mathfrak{M}$ if and only if (b, a') is contained in $\mathfrak{M}$.

$M_2(a, a')$. If (a, b') and (b, a') are contained in $\mathfrak{M}$, then (b, b') is contained in $\mathfrak{M}$.

μ is said to be weakly classable at (a, a') if and only if μ satisfies the condition $M_1(a, a')$. μ is said to be locally classable at (a, a') if and only if μ satisfies the conditions $M_1(a, a')$ and $M_2(a, a')$. μ is said to be classable if and only if μ satisfies the condition $M_2(a, a')$ for every element (a, a') in $\mathfrak{M}$.

Lemma 1. *Let $\mathfrak{A}$ be an algebraic system, and let θ and φ be congruences on $\mathfrak{A}$. And let μ be the natural meromorphism between the residue class systems $\mathfrak{A}/\theta$ and $\mathfrak{A}/\varphi$, i.e. the meromorphism be-*

tween $\mathfrak{A}/\theta$ *and* $\mathfrak{A}/\varphi$ *which is defined by the graph* $\mathfrak{M}=\{((x\mid\theta), (x\mid\varphi))\mid x\in\mathfrak{A}\}$.[2] *And let* a *be an element of* $\mathfrak{A}$. *Then*

(1) θ *and* φ *are weakly permutable at* a *if and only if* μ *is weakly classable at* $((a\mid\theta), (a\mid\varphi))$.

(2) θ *and* φ *are locally permutable at* a *if and only if* μ *is locally classable at* $((a\mid\theta), (a\mid\varphi))$.

Proof. It is sufficient to prove the following two propositions:

(i) θ and φ satisfy the condition $K_1(a)$ if and only if μ satisfies the condition $M_1((a\mid\theta), (a\mid\varphi))$.

(ii) θ and φ satisfy the condition $K_2(a)$ if and only if μ satisfies the condition $M_2((a\mid\theta), (a\mid\varphi))$.

And these propositions can be easily verified from the definitions.

Lemma 2. *Let* A *be a set of composition-identities with respect to a system of compositions, and let* μ *be a meromorphism between* A*-algebraic systems* $\mathfrak{A}$ *and* $\mathfrak{B}$. *Then there exist an* A*-algebraic system* $\mathfrak{N}$ *and congruences* θ *and* φ *on* $\mathfrak{N}$ *such that* $\mathfrak{A}\cong\mathfrak{N}/\theta$ *and* $\mathfrak{B}\cong\mathfrak{N}/\varphi$, *and that* $a\overset{\mu}{\longleftrightarrow}b$ *if and only if* $a(\theta)\overset{\nu}{\longleftrightarrow}b(\varphi)$, *where* $a(\theta)$ *is the congruence class of* $\mathfrak{N}/\theta$ *corresponding to* a *by the above first isomorphism,* $b(\varphi)$ *is the congruence class of* $\mathfrak{N}/\varphi$ *corresponding to* b *by the above second isomorphism and* ν *is the natural meromorphism between* $\mathfrak{N}/\theta$ *and* $\mathfrak{N}/\varphi$.

Proof. Let $\mathfrak{N}$ be the subsystem of the direct product $\mathfrak{A}\times\mathfrak{B}$ which is the graph of the meromorphism μ. Then $\mathfrak{N}$ is an A-algebraic system. Now let θ be the congruence on $\mathfrak{N}$ defined by

$$(a_1, b_1)\overset{\theta}{\sim}(a_2, b_2) \text{ if and only if } a_1=a_2,$$

and let φ be the congruence on $\mathfrak{N}$ defined by

$$(a_1, b_1)\overset{\varphi}{\sim}(a_2, b_2) \text{ if and only if } b_1=b_2.$$

Then it is clear that the mapping

$$a\longrightarrow a(\theta)=\{(a, x)\mid x\in\mathfrak{B} \text{ and } (a, x)\in\mathfrak{N}\}$$

is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{N}/\theta$, and the mapping

$$b\longrightarrow b(\varphi)=\{(y, b)\mid y\in\mathfrak{A} \text{ and } (y, b)\in\mathfrak{N}\}$$

is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{N}/\varphi$. Hence we have that

$$(a, b)\in\mathfrak{N} \text{ if and only if } a(\theta)\cap b(\varphi)\neq\phi,$$

i. e.

$$a\overset{\mu}{\longleftrightarrow}b \text{ if and only if } a(\theta)\overset{\nu}{\longleftrightarrow}b(\varphi).$$

This completes the proof of this lemma.

The following two lemmas can be easily obtained from Lemmas 1 and 2.

Lemma 3. *Let* V *be a system of compositions which contains a* 0*-ary composition* e. *And let* A *be a set of composition-identities with respect to* V. *If any two congruences on any* A*-algebraic*

2) $(x\mid\theta)$ denotes the congruence class of $\mathfrak{A}$ modulo θ which contains x.

system are weakly permutable at e,[3] *then any meromorphism between any two* A-*algebraic systems is weakly classable at* (e, e).

Lemma 4. *Under the same condition of Lemma* 3, *if any two congruences of any* A-*algebraic system are locally permutable at* e, *then any meromorphism between any two* A-*algebraic systems is locally classable at* (e, e).

Lemma 5. *Let* V *be a system of compositions which contains a* 0-*ary composition* e, *and let* A *be a set of composition-identities with respect to* V. *If any meromorphism between any two* A-*algebraic systems satisfies the condition* $M_1(e, e)$, *then there exists a derived composition* $f(\xi, \eta)$ *of* V *such that*

$$f(\xi, \xi)\overset{A}{=}e \text{ and } f(\xi, e)\overset{A}{=}\xi.$$

Proof. Let $\mathfrak{A}$ be the free A-algebraic system generated by the free system of only one generator b, and let $\mathfrak{A}'$ be the free A-algebraic system generated by the free system of only one generator b'. Now let $\mathfrak{M}$ be the subsystem of the direct product $\mathfrak{A}\times\mathfrak{A}'$ which is generated by $\{(b, b'), (b, e)\}$. Then it is clear that $\mathfrak{M}$ gives the meromorphism μ between $\mathfrak{A}$ and $\mathfrak{A}'$. By the assumption of this theorem, μ satisfies the condition $M_1(e, e)$. Hence we have $(e, b')\in\mathfrak{M}$, because (e, e), (b, b'), and (b, e) are contained in $\mathfrak{M}$. Hence there exists a derived composition $f(\xi, \eta)$ of V such that

$$f((b, b'), (b, e))=(e, b').$$

Hence we have

$$f(b, b)=e \text{ and } f(b', e)=b'.$$

Since $\{b\}$ and $\{b'\}$ are free systems of generators, we have

$$f(\xi, \xi)\overset{A}{=}e \text{ and } f(\xi, e)\overset{A}{=}\xi.$$

Theorem 1. *Let* V *be a system of compositions which contains a* 0-*ary composition* e, *and let* A *be a set of composition-identities with respect to* V. *Then the following three conditions are equivalent*:

A-1. *Any two congruences on any* A-*algebraic system are weakly permutable at* e.

B-1. *Any meromorphism between any two* A-*algebraic systems is weakly classable at* (e, e).

C-1. *There exists a derived composition* $f(\xi, \eta)$ *of* V *such that*

$$f(\xi, \xi)\overset{A}{=}e \text{ and } f(\xi, e)\overset{A}{=}\xi.$$

Proof. By Lemma 3, it is clear that the condition A-1 implies B-1. It follows from Lemma 5 that the condition B-1 implies C-1.

Now we shall prove that the condition C-1 implies A-1. Let

3) The 0-ary composition e can be considered as a constant element in the A-algebraic system.

$\mathfrak{A}$ be any A-algebraic system, and let θ and φ be any two congruences on $\mathfrak{A}$. Now let x and y be elements in $\mathfrak{A}$ such that $e \overset{\theta}{\sim} y \overset{\varphi}{\sim} x$. Then by C-1, we have

$$e = f(x, x) \overset{\varphi}{\sim} f(x, y) \overset{\theta}{\sim} f(x, e) = x.$$

Hence there exists an element $z(=f(x, y))$ such that $e \overset{\theta}{\sim} z \overset{\varphi}{\sim} x$. Conversely, if $e \overset{\varphi}{\sim} z \overset{\theta}{\sim} x$, then we have similarly that there exists an element y such that $e \overset{\theta}{\sim} y \overset{\varphi}{\sim} x$. Hence we have the condition A-1. This completes the proof.

Lemma 6. *Let V be a system of compositions which contains a 0-ary composition e, and let A be a set of composition-identities with respect to V. If any meromorphism between any two A-algebraic systems satisfies the condition $M_2(e, e)$, then there exists a derived composition $g(\xi, \eta)$ of V such that*

$$g(\xi, e) \overset{A}{=} \xi \text{ and } g(e, \xi) \overset{A}{=} \xi.$$

Proof. Let $\mathfrak{A}$ be the free A-algebraic system generated by the free system of only one generator b, and let $\mathfrak{A}'$ be the free A-algebraic system generated by the free system of only one generator b'. Now let $\mathfrak{M}$ be the subsystem of the direct product $\mathfrak{A} \times \mathfrak{A}'$ which is generated by $\{(b, e), (e, b')\}$. Then it is clear that $\mathfrak{M}$ gives the meromorphism μ between $\mathfrak{A}$ and $\mathfrak{A}'$. By the assumption of this theorem, μ satisfies the condition $M_2(e, e)$. Hence we have $(b, b') \in \mathfrak{M}$, because (e, e), (b, e), and (e, b') are contained in $\mathfrak{M}$. Hence there exists a derived composition $g(\xi, \eta)$ of V such that

$$g((b, e), (e, b')) = (b, b').$$

Hence we have

$$g(b, e) = b \text{ and } g(e, b') = b'.$$

Since $\{b\}$ and $\{b'\}$ are free systems of generators, we have

$$g(\xi, e) \overset{A}{=} \xi \text{ and } g(e, \xi) \overset{A}{=} \xi.$$

Theorem 2. *Let V be a system of compositions which contains a 0-ary composition e, and let A be a set of composition-identities with respect to V. Then the following three conditions are equivalent:*

A-2. *Any two congruences on any A-algebraic system are locally permutable at e.*

B-2. *Any meromorphism between any two A-algebraic systems is locally classable at (e, e).*

C-2. *There exist derived compositions $f(\xi, \eta)$ and $g(\xi, \eta)$ of V such that*

$$f(\xi, \xi) \overset{A}{=} e,\ f(\xi, e) \overset{A}{=} \xi \text{ and } g(\xi, e) \overset{A}{=} \xi,\ g(e, \xi) \overset{A}{=} \xi.$$

Proof. By Lemma 4, it is clear that the condition A-2 implies B-2. It follows from Lemmas 5 and 6 that the condition B-2 implies C-2.

Now we shall prove that the condition C-2 implies A-2. It is clear that the condition C-2 implies C-1. And by Theorem 1, the condition C-1 implies A-1. Hence C-2 implies A-1, i. e. any two congruences θ and φ on any A-algebraic system satisfy the condition $K_1(e)$. Hence it is sufficient to prove that any two congruences θ and φ on any A-algebraic system satisfy the condition $K_2(e)$. Now let $\mathfrak{A}$ be any A-algebraic system, and let θ and φ be any two congruences on $\mathfrak{A}$. And let x and y be elements in $\mathfrak{A}$ such that $x \overset{\theta}{\sim} e \overset{\varphi}{\sim} y$. Then by the condition C-2, we have

$$x=g(x, e) \overset{\varphi}{\sim} g(x, y) \overset{\theta}{\sim} g(e, y)=y.$$

Hence there exists an element $z(=g(x, y))$ such that $x \overset{\varphi}{\sim} z \overset{\theta}{\sim} y$. Hence θ and φ satisfy the condition $K_2(e)$. This completes the proof.

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