168. *Special Type of Separable Algebra over a Commutative Ring*

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In the previous paper [4], we considered a type of separable algebra over a field which has the simple ideal components whose degrees are all prime to the characteristic of the field. In this paper we consider the case of algebra over a commutative ring.

Let Λ be an algebra over a commutative ring R. In the enveloping algebra $\Lambda^e=\Lambda\otimes_R\Lambda^0$ we consider the involution $*$ defined by $(x\otimes y^0)^*=y\otimes x^0$ for $x\otimes y^0\in\Lambda^e$. We set $J=\{x\otimes 1^0-1\otimes x^0 \mid x\in\Lambda\}$, then $J^*=J$. Let A be the right annihilator of J in Λ^e, then A^* is the left annihilator of J and a left ideal in Λ^e. Let $\varphi:\Lambda^e\rightarrow\Lambda$ be the Λ^e-homomorphism defined by $\varphi(x\otimes y^0)=xy$, then $\varphi(A^*)$ is a two sided ideal of Λ. In this paper we shall call Λ a strongly separable algebra over R when $\varphi(A^*)=\Lambda$.

In § 1, we shall show that Λ is a strongly separable algebra over R if and only if Λ is a separable algebra over R and $\Lambda=C\oplus[\Lambda,\Lambda]$ where C is the center of Λ and $[\Lambda,\Lambda]$ is the C-submodule of Λ generated by $xy-yx$ for all $x,y\in\Lambda$. In § 2, we consider an R-algebra Λ such that Λ is an R-projective module, and we shall show that if $A\neq 0$ then there exists a non zero left ideal in Λ which is generated by a finite number of elements as R-module. Finally, we have that for a central separable R-algebra Λ, Λ is hereditary if and only if R is hereditary. In this paper we assume that every rings and algebras have identity elements.

1. Strongly separable algebra.

PROPOSITION 1. *Let Λ be an algebra over R. Then $\varphi(A^*)=\Lambda$ if and only if $\Lambda^e=\Lambda^eJ\oplus A^*$. If $\varphi(A^*)=\Lambda$ then Λ is a separable algebra over R and $\Lambda=C\oplus[\Lambda,\Lambda]$, where C is the center of Λ and $[\Lambda,\Lambda]$ is the C-submodule of Λ generated by $xy-yx$ for all $x,y\in\Lambda$.*

Proof. If $\Lambda^e=\Lambda^eJ\oplus A^*$ then we have $\varphi(A^*)=\Lambda$. Now we assume $\varphi(A^*)=\Lambda$. Since $\operatorname{Ker}\varphi=\Lambda^eJ$, we have $\Lambda^e=A^*+\Lambda^eJ$. Therefore we have $\Lambda^{e*}=A^{**}+J^*\cdot\Lambda^{e*}$ and $\Lambda^e=A+J\Lambda^e$. Let $1\otimes 1^0=z_1+z_2$ with $z_1\in A$, $z_2\in J\Lambda^e$. If $x\in A^*\cap\Lambda^eJ$ then $x=x\cdot 1\otimes 1^0=xz_1+xz_2=0$. It follows that $A^*\cap\Lambda^eJ=0$ and $\Lambda^e=A^*\oplus\Lambda^eJ$. Thus the first half of the proposition is proved. If $\varphi(A^*)=\Lambda$, then φ induces an isomorphism of A^* onto Λ therefore Λ is a separable algebra over R. Since $\Lambda^e=\Lambda^eJ\oplus A^*$, there are orthogonal idempotents $e_1\in\Lambda^eJ$ and

$e_2 \in A^*$ such that $1\otimes 1^0 = e_1 + e_2$.[1] Then $\Lambda^e J = \Lambda^e e_1$ $A^* = \Lambda^e e_2$ and $\varphi(e_2) = 1$. Since A is a right ideal, $A = Ae_1 + Ae_2$, where $Ae_1 = A\Lambda^e e_1 = A\Lambda^e J = AJ$ and $Ae_2 \subseteqq AA^* \subseteqq A \cap A^*$. Now $Ae_1 \cap (A \cap A^*) \subseteqq \Lambda^e J \cap A^* = 0$ therefore we have $Ae_2 = A \cap A^*$ and $A = AJ \oplus (A \cap A^*)$. Taking *, we have $A^* = JA^* \oplus (A \cap A^*)$. Since φ is an isomorphism of A^* and Λ, $\Lambda = \varphi(JA^*) \oplus \varphi(A \cap A^*)$. Now $\varphi(A \cap A^*) = \varphi(Ae_2) = \varphi(A) = C$ by [1], Proposition 1.1, and $\varphi(JA^*) = [\Lambda, \Lambda]$ as shown in [4], therefore we have $\Lambda = [\Lambda, \Lambda] \oplus C$.

LEMMA 1. *Let Λ be an algebra over R, and C the center of Λ. Then Λ is a strongly separable algebra over R if and only if Λ is a strongly separable algebra over C and C is a separable algebra over R.*

Proof. Suppose that Λ is a strongly separable algebra over R. Let A_C be the right annihilator of $\{x\otimes_C 1^0 - 1\otimes_C x^0 \in \Lambda\otimes_C\Lambda^0 \mid x \in \Lambda\}$ in $\Lambda\otimes_C\Lambda^0$ and ψ: $\Lambda\otimes_R\Lambda^0 \to \Lambda\otimes_C\Lambda^0$ the ring homomorphism defined by $\psi(x\otimes y^0) = x\otimes_C y^0$. Then $\psi(A^*) \subseteqq A_C^*$ and we have a commutative diagram

$$\begin{array}{ccc} \Lambda\otimes_R\Lambda^0 & \xrightarrow{\ \psi\ } & \Lambda\otimes_C\Lambda^0 \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & \Lambda & \end{array}$$

where φ' is defined by $\varphi'(x\otimes_C y^0) = xy$. Since $\varphi(A^*) = \varphi'(\psi(A^*)) \subseteqq \varphi'(A_C^*)$ and $\varphi(A^*) = \Lambda$ by assumption, we have $\varphi'(A_C^*) = \Lambda$ and Λ is a strongly separable algebra over C. By Proposition 1 Λ is separable over R, therefore C is also separable over R by [1], Theorem 2.3. Conversely assume that Λ is a strongly separable algebra over C and C is a separable algebra over R. Since the exact sequence $0 \to \mathrm{Ker}\,\varphi'' \to C\otimes_R C \xrightarrow{\varphi''} C \to 0$ splits where φ'' is defined by $\varphi''(x\otimes y^0) = x\cdot y$, the sequence

$$(\Lambda\otimes_R\Lambda^0)\otimes_{C\otimes_R C}(C\otimes_R C) \longrightarrow (\Lambda\otimes_R\Lambda^0)\otimes_{C\otimes_R C} C \longrightarrow 0$$

splits, therefore $\Lambda\otimes_R\Lambda^0 \xrightarrow{\psi} \Lambda\otimes_C\Lambda^0 \to 0$ splits. There exists a homomorphism ξ: $\Lambda\otimes_C\Lambda^0 \to \Lambda\otimes_R\Lambda^0$ such that $\psi\circ\xi = $ identity. We denote by A' the annihilator of $\{x\otimes 1^0 - 1\otimes x^0 \in C\otimes_R C \mid x \in C\}$ in $C\otimes_R C$. Since C is separable over R, there exists z in A' such that $z^* = z$, $z^2 = z$, and $\varphi''(z) = 1$. Since Λ is strongly separable over C, there exists an idempotent element e_1 in A_C^* such that $e_1^* = e_1$ and $\varphi'(e_1) = 1$. Let η: $C\otimes_R C \to \Lambda\otimes_R\Lambda$ be the homomorphism induced by the inclusion $C \to \Lambda$ and let $e = \eta(z)\cdot\xi(e_1)$. Then we have $e^* = e$ and $\varphi(e) = \varphi(\eta(z)\cdot\xi(e_1)) = \varphi''(z)\cdot\varphi'(e_1) = 1$. Moreover e is contained in A^*. Therefore we have $\varphi(A^*) = \Lambda$.

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1) e_1 and e_2 are symmetric, i.e. $e_1^* = e_1$ and $e_2^* = e_2$, because $e_2^* e_1 = e_2^* - e_2^* e_2 \in \Lambda^e J \cap A^* = 0$ and $e_2^* = e_2^* e_2$, and $e_2 = (e_2^* e_2)^* = e_2^* e_2 = e_2^*$.

LEMMA 2. *Suppose that the center C of a ring Λ is a field. Then Λ is a strongly separable algebra over C if and only if Λ is separable over C and $\Lambda=C\oplus[\Lambda,\Lambda]$.*

Proof. The "only if" part is proved in Proposition 1. To prove the "if" part we may assume that C is an algebraically closed field by [4], Lemma 3. Then a separable algebra Λ over C is a direct sum of total matrix rings over C, therefore we may assume that $\Lambda=C_n$. Now since

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}-\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

we have

$$\begin{pmatrix}1&&&0\\&1&&\\&&\ddots&\\0&&&1\end{pmatrix}-\left\{\left(\begin{array}{c|c}\begin{matrix}1&1\\0&1\end{matrix}&0\\\hline 0&0\end{array}\right)\left(\begin{array}{c|c}\begin{matrix}1&0\\1&1\end{matrix}&0\\\hline 0&0\end{array}\right)-\left(\begin{array}{c|c}\begin{matrix}1&0\\1&1\end{matrix}&0\\\hline 0&0\end{array}\right)\left(\begin{array}{c|c}\begin{matrix}1&1\\0&1\end{matrix}&0\\\hline 0&0\end{array}\right)\right\}=\begin{pmatrix}0&&&&0\\&2&&&\\&&1&&\\&&&\ddots&\\0&&&&1\end{pmatrix},$$

and repeating the same argument we have

$$\begin{pmatrix}1&&&0\\&1&&\\&&\ddots&\\0&&&1\end{pmatrix}-\begin{pmatrix}0&&&&0\\&0&&&\\&&\ddots&&\\&&&0&\\0&&&&n\end{pmatrix}\in[\Lambda,\Lambda].$$

Therefore if n is a multiple of the characteristic p of C, then the unit matrix E is contained in $[\Lambda,\Lambda]$. This contradicts the assumption $C\cap[\Lambda,\Lambda]=0$. Thus n is prime to p, and then by the Theorem in [4] Λ is strongly separable C-algebra.

LEMMA 3. *Let Λ be the center of a ring Λ. Then Λ is a strongly separable algebra over C if and only if Λ is separable over C and $\Lambda=C\oplus[\Lambda,\Lambda]$.*

Proof. The "only if" part is proved in Proposition 1. We shall prove here the "if" part. Let $\mathfrak{m}$ be any maximal ideal of C. For the localization $C_{\mathfrak{m}}$ of C by $\mathfrak{m}$, we set $\Lambda_{\mathfrak{m}}=\Lambda\otimes_C C_{\mathfrak{m}}$ and the right annihilator of $\{x\otimes 1^0-1\otimes x^0\in\Lambda^e_{\mathfrak{m}}\mid x\in\Lambda_{\mathfrak{m}}\}$ in $\Lambda^e_{\mathfrak{m}}=\Lambda_{\mathfrak{m}}\otimes_{C_{\mathfrak{m}}}\Lambda^0_{\mathfrak{m}}$ is denoted by $A_{\mathfrak{m}}$. If an element of Λ is identified with the image by the homomorphism $\Lambda\to\Lambda\otimes_C C_{\mathfrak{m}}$, then we have $A^*_{\mathfrak{m}}=A^*C_{\mathfrak{m}}$ and $\varphi_{\mathfrak{m}}(A^*_{\mathfrak{m}})=\varphi(A^*)C_{\mathfrak{m}}$ where $\varphi_{\mathfrak{m}}$: $\Lambda_{\mathfrak{m}}\otimes_{C_{\mathfrak{m}}}\Lambda^0_{\mathfrak{m}}\to\Lambda_{\mathfrak{m}}$. Consequently, we have that $\varphi(A^*)\ni 1$ if and only if $\varphi_{\mathfrak{m}}(A^*_{\mathfrak{m}})\ni 1$ for all maximal ideals $\mathfrak{m}$ of C. On the other hand if Λ is a central separable C-algebra and $\Lambda=$

$C\oplus[\Lambda, \Lambda]$, then we have that $\Lambda_{\mathfrak{m}}$ is a central separable $C_{\mathfrak{m}}$-algebra and $\Lambda_{\mathfrak{m}}=C_{\mathfrak{m}}\oplus[\Lambda_{\mathfrak{m}}, \Lambda_{\mathfrak{m}}]$. Therefore we may assume that C is a local ring with the maximal ideal $\mathfrak{m}$. By proposition 1.4 and Corollary 1.6 in [1] we have that $\Lambda/\mathfrak{m}\Lambda\cong\Lambda\otimes_C C/\mathfrak{m}$ is a central separable $C/\mathfrak{m}$-algebra and $\bar{\Lambda}=\bar{C}+[\bar{\Lambda}, \bar{\Lambda}]$ if we set $\bar{\Lambda}=\Lambda/\mathfrak{m}\Lambda$ and $\bar{C}=C+\mathfrak{m}\Lambda/\mathfrak{m}\Lambda\cong C/\mathfrak{m}$. If $x\in\bar{C}\cap[\bar{\Lambda}, \bar{\Lambda}]$ then $x=c+\mu=\lambda+\mu'$, $c\in C$, $\lambda\in[\Lambda, \Lambda]$, $\mu, \mu'\in\mathfrak{m}\Lambda$. Then $c-\lambda\in\mathfrak{m}\Lambda=\mathfrak{m}C\oplus\mathfrak{m}[\Lambda, \Lambda]$, therefore $c\in\mathfrak{m}C$, $\lambda\in\mathfrak{m}[\Lambda, \Lambda]$. It follows that $\bar{C}\cap[\bar{\Lambda}, \bar{\Lambda}]=0$ and $\bar{\Lambda}=\bar{C}\oplus[\bar{\Lambda}, \bar{\Lambda}]$. By Lemma 2 $\bar{\Lambda}$ is a strongly separable $\bar{C}$-algebra. Accordingly, we have $\bar{\Lambda}^e=\bar{A}^*\oplus\bar{\Lambda}^e\bar{J}$ where $\bar{A}$ is the right annihilator of $\bar{J}=\{\bar{x}\otimes\bar{1}^0-\bar{1}\otimes\bar{x}^0\in\bar{\Lambda}^e \mid \bar{x}\in\bar{\Lambda}\}$ in $\bar{\Lambda}^e$. Since $A=\mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda^e)$ (see [1], p. 369) and Λ is a projective Λ^e-module, we have

$$A\otimes_C C/\mathfrak{m}\cong\mathrm{Hom}_{\Lambda^e\otimes_C C/\mathfrak{m}}(\Lambda\otimes C/\mathfrak{m}\Lambda^e\otimes_C C/\mathfrak{m})=\mathrm{Hom}_{\bar{\Lambda}^e}(\bar{\Lambda}, \bar{\Lambda}^e).$$

Therefore $A\otimes_C C/\mathfrak{m}=\bar{A}$ and $A^*\otimes_C C/\mathfrak{m}=\bar{A}^*$. It follows that $\Lambda^e=A^*+\Lambda^e J+\mathfrak{m}\Lambda^e$. Since $\mathfrak{m}\Lambda^e$ is contained in the radical of Λ^e, by Nakayama's Lemma we have $\Lambda^e=A^*+\Lambda^e J$ and then $\varphi(A^*)=\Lambda$.

By Lemmas 1 and 3, we have

THEOREM 1. *Let Λ be an algebra over an arbitrary commutative ring R and C the center of Λ. Then Λ is a strongly separable algebra over R if and only if Λ is a separable algebra over R and $\Lambda=C\oplus[\Lambda, \Lambda]$.*

From the proof of Lemma 3 we have

COROLLARY 1. *Λ is a strongly separable R-algebra if and only if $\Lambda/\mathfrak{m}\Lambda$ is a strongly separable $R/\mathfrak{m}$-algebra for all maximal ideals $\mathfrak{m}$ of R.*

COROLLARY 2. *If Λ_1 and Λ_2 are strongly separable R-algebras then $\Lambda_1\otimes_R\Lambda_2$ is either 0 or a strongly separable R-algebra.*

2. Annihilator ideal A. Let Λ be an R-algebra such that Λ is projective as R-module. Then there exists a family $\{\varphi_\kappa, \lambda_\kappa\}_{\kappa\in I}$ of homomorphisms φ_κ in $\mathrm{Hom}_R(\Lambda, R)$ and elements λ_κ in Λ such that $x=\sum_\kappa \varphi_\kappa(x)\lambda_\kappa$ for any element x in Λ. In this section, we consider the right annihilator A of $J=\{x\otimes 1^0-1\otimes x^0\in\Lambda^e \mid x\in\Lambda\}$ in $\Lambda^e=\Lambda\otimes_R\Lambda^0$ for such an algebra Λ. We can see $\mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda, R), \Lambda)$ as Λ^e-right module by setting $f\cdot x\otimes y^0(g)=y\cdot f(x\cdot g)$ for $x\otimes y^0\in\Lambda^e$, $f\in\mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda, R), \Lambda)$ and $g\in\mathrm{Hom}_R(\Lambda R)$ where $x\cdot g(z)=g(z\cdot x)$, $z\in\Lambda$.

LEMMA 4 (cf. [2], VI, Proposition 5.2). *If Λ is an R-algebra which is projective as R-module, then the homomorphism θ: $\Lambda\otimes_R\Lambda^0\to\mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda, R), \Lambda)$ defined by $\theta(x\otimes y_0)(f)=f(x)\cdot y$ is a Λ^e-monomorphism, and $\theta(A)$ is contained in $\mathrm{Hom}_\Lambda^r(\mathrm{Hom}_R(\Lambda, R), \Lambda)$ where $\mathrm{Hom}_R(\Lambda, R)$ is regarded as Λ-right module by setting $f\cdot\lambda(z)=f(\lambda\cdot z)$ for $f\in\mathrm{Hom}_R(\Lambda, R)$.*

Proof. Let f be an element of $\mathrm{Hom}_R(\Lambda, R)$. Since

$$\theta(x\otimes y^0\cdot x_1\otimes y_1^0)(f)=\theta(xx_1\otimes(y_1y)^0)(f)=f(xx_1)y_1y=(x_1f)(x)\cdot y_1y$$
$$=y_1\cdot(x_1\cdot f)(x)\cdot yy_1(\theta(x\otimes y^0)(x_1f))=\theta(x\otimes y^0)\cdot x_1\otimes y_1^0(f),$$

θ is a Λ^e-homomorphism. If $\theta\Big(\sum_i x_i\otimes y_i^0\Big)=0$ for an element $\sum_i x_i\otimes y_i^0$ in $\Lambda\otimes_R\Lambda^0$, then $\sum_i f(x_i)y_i=0$ for every element f in $\mathrm{Hom}_R(\Lambda, R)$, therefore we have $\sum_i x_i\otimes y_i^0=\sum_i\sum_\kappa \varphi_\kappa(x_i)\lambda_\kappa\otimes y_i^0=\sum_i\sum_\kappa \lambda_\kappa\otimes\varphi_\kappa(x_i)y_i=0$ by using the above family $\{\varphi_\kappa, \lambda_\kappa\}_{\kappa\in I}$. Hence θ is a Λ^e-monomorphism. Let $\sum_i x_i\otimes y_i^0$ be any element in A. Then we have $\sum_i \lambda x_i\otimes y_i^0=\sum_i x_i\otimes(y_i\lambda)^0$ for every λ in Λ. Set $\psi=\theta\Big(\sum_i x_i\otimes y_i^0\Big)$, then we have

$$\psi(f\lambda)=\sum_i f\cdot\lambda(x_i)\cdot y_i=\sum_i f(\lambda x_i)y_i=\theta\Big(\sum_i \lambda x_i\otimes y_i^0\Big)(f)$$
$$=\theta\Big(\sum_i x_i\otimes(y_i\lambda)^0\Big)(f)=\sum_i f(x_i)\cdot y_i\lambda=\psi(f)\lambda.$$

Therefore $\psi\in\mathrm{Hom}_\Lambda^r(\mathrm{Hom}_R(\Lambda, R), \Lambda)$.

THEOREM 2. *Let Λ be an algebra over R such that Λ is an R-projective module. If $A\neq 0$ then there exists a right ideal of Λ which is a finitely generated R-module.*

Proof. For $\sum_i x_i\otimes y_i^0\neq 0$ in A we set $z_\kappa=\theta\Big(\sum_i x_i\otimes y_i^0\Big)(\varphi_\kappa)=\sum_i \varphi_\kappa(x_i)y_i$, where $\{\varphi_\kappa, \lambda_\kappa\}$ is the family as above. Since $\theta\Big(\sum_i x_i\otimes y_i^0\Big)\neq 0$, there is a non zero element z_κ and the number of $z_\kappa\neq 0$ is finite. For every element λ in Λ, we have

$$z_\kappa\lambda=\theta\Big(\sum_i x_i\otimes y_i^0\Big)(\varphi_\kappa)\cdot\lambda=\theta(\sum x_i\otimes y_i^0)(\varphi_\kappa\cdot\lambda)=\sum_i \varphi_\kappa(\lambda x_i)y_i$$
$$=\sum_i \varphi_\kappa\Big(\lambda\sum_j \varphi_j(x_i)\lambda_j\Big)y_i=\sum_{ij}\varphi_\kappa(\lambda\lambda_j)\varphi_j(x_i)y_i=\sum_j \varphi_\kappa(\lambda\lambda_j)z_j,$$

and $\sum_j \varphi_\kappa(\lambda\lambda_j)z_j$ is contained in $\sum_\kappa Rz_\kappa$. It follows that $\mathfrak{A}=\sum_\kappa Rz_\kappa$ is a right ideal of Λ which is a finitely generated R-module.

REMARK. If Λ is an R-algebra which is a finitely generated projective R-module, then $\theta: \Lambda\otimes_R\Lambda^0\to\mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda, R), \Lambda)$ is an isomorphism ([2], VI, Proposition 5.2). Then we have $\theta(A^*)=\mathrm{Hom}_\Lambda^l(\mathrm{Hom}_R(\Lambda, R), \Lambda)$ and $\theta(A)=\mathrm{Hom}_\Lambda^r(\mathrm{Hom}_R(\Lambda, R), \Lambda)$. For the family $\{\varphi_\kappa, \lambda_\kappa\}_{\kappa\in I}$, if we set $\mathrm{Tr}=\sum_i \lambda_i\varphi_i$, then Tr is contained in $\mathrm{Hom}_R(\Lambda, R)$ and we have $\varphi(A^*)=\{f(\mathrm{Tr})\mid f\in\mathrm{Hom}_\Lambda^l(\mathrm{Hom}_R(\Lambda, R), \Lambda)\}$ where $\varphi: \Lambda\otimes_R\Lambda^0\to\Lambda$.

PROPOSITION 2. *Let Λ be a central separable R-algebra. Then Λ is hereditary* (*see* [2], *p.* 13) *if and only if R is hereditary.*

Proof. If Λ is R-separable then Λ is R-semisimple in the sense of Hattori [3]. By [3], § 2, p. 408, we have that if R is hereditary then Λ is hereditary. Conversely, we suppose that Λ is hereditary. For any ideal $\mathfrak{a}$ of R, $\mathfrak{a}\Lambda$ is a projective Λ-module. For the exact sequence $0\to\mathfrak{a}\to R\to R/\mathfrak{a}\to 0$, we have an exact sequence

$$0\longrightarrow\mathfrak{a}\otimes_R\Lambda\longrightarrow R\otimes_R\Lambda\longrightarrow R/\mathfrak{a}\otimes_R\Lambda\longrightarrow 0$$

since Λ is R-projective. Therefore we have $\mathfrak{a}\otimes_R\Lambda\cong\mathfrak{a}\cdot\Lambda$. Since Λ is R-projective and R is a direct summand of Λ as R-module, we have that $\mathfrak{a}$ is a direct summand of R-projective module $\mathfrak{a}\cdot\Lambda\cong\mathfrak{a}\otimes_R\Lambda$ as R-module, therefore $\mathfrak{a}$ is R-projective. Thus R is hereditary.

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