13. A Priori Estimates for Certain Differential Operators with Double Characteristics

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1. Introduction. In the study of the uniqueness of the Cauchy problem for linear partial differential operators with non-analytic coefficients, Carleman [1] proved a priori estimates of the form

$$\tau^{\gamma}\int |D^{\alpha}u|^2 e^{2\tau\varphi}dx \leqq K\int |P(x, D)u|^2 e^{2\tau\varphi}dx, \quad u\in C_0^{\infty}(\Omega), \tau>\tau_0, \tag{1.1}$$

where K and γ are constants which are independent of u and τ, and φ is a fixed function. For operators with simple characteristics, many uniqueness theorems have been proved, and it is also known that there exist operators with double complex characteristics for which uniqueness theorems are proved (see Pedersen [8], Mizohata [7], Hörmander [2], Shirota [9], Malgrange [6], Kumano-go [4]).

In this note we shall give a priori estimates of the form (1.1) for the operators of the form

$$P(x, \xi)=P^1(x, \xi)P^2(x, \xi), \tag{1.2}$$

where P^1 is principally normal and P^2 is elliptic (definitions are given in section 2).

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2. Notations, definitions and theorems from Hörmander. Let α be n-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ of non-negative integers and we shall use the notations:

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n,\ |x|^2=x_1^2+x_2^2+\cdots+x_n^2,\ \xi^{\alpha}=\xi_1^{\alpha_1}\cdot\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n},$$

$$D_j=-i\frac{\partial}{\partial x_j}, \text{ where } i=\sqrt{-1} \text{ and } D^{\alpha}=D_1^{\alpha_1}\cdot D_2^{\alpha_2}\cdots D_n^{\alpha_n}.$$

Let $P(x, \xi)$ be a polynomial of degree m in n variabils $\xi_1, \xi_2, \cdots, \xi_n$ whose coefficients are functions of x, and denote by $P(x, D)$ the differential operator obtained if ξ_j is replaced by D_j. Let $P_m(x, \xi)$ be the principal part of $P(x, \xi)$ (the homogeneous part of degree m) and we shall also use the notations:

$$P_m^{(j)}(x, \xi)=\frac{\partial}{\partial \xi_j}P_m(x, \xi), \quad P_{m,j}(x, \xi)=\frac{\partial}{\partial x_j}P_m(x, \xi).$$

The following definitions and theorems in this section are quoted from Hörmander [3], Chap. 8.

Definition 1. We shall say that $P(x, D)$ is principally normal in $\bar{\Omega}$ if the coefficients of P_m are in $C^1(\bar{\Omega})$ and there exists a differential operator $Q_{m-1}(x, D)$, homogeneous of degree $m-1$ in D, with coefficients in $C^1(\bar{\Omega})$, such that

(2.1) $$2\mathrm{I}_m \sum P_{m,j}(x, \xi)\overline{P_m^{(j)}(x, \xi)} = 2ReP_m(x, \xi)\overline{Q_{m-1}(x, \xi)}, \ \xi \in R_n.$$

Definition 2. Let P be either elliptic or principally normal. The oriented surface defined by

(2.2) $$\psi(x) = \psi(x^0)$$

will be called pseudo-convex with respect to P at the point x if

(2.3) $$\sum_{j,k} \frac{\partial^2 \psi}{\partial x_j \partial x_k} P_m^{(j)}(x, \xi)\bar{P}_m^{(k)}(x, \xi) + \\ + Re \sum_{j\ k} (P_m^{(j)}{}_k(x, \xi)\bar{P}_m^{(k)}(x, \xi) - P_{m,k}(x, \xi)\bar{P}_m^{(k,j)}(x, \xi)) \frac{\partial \psi}{\partial x_j} > 0$$

for all $\xi \neq 0$ in R_n, satisfying the equations

(2.4) $$P_m(x, \xi) = 0, \quad \sum P_m^{(j)}(x, \xi) \frac{\partial \psi}{\partial x_j} = 0.$$

The surface is called strongly pseudo-convex with respect to P at the point x, if, in addition,

(2.5) $$\sum_{j,k} \frac{\partial^2 \psi}{\partial x_j \partial x_k} P_m^{(j)}(x, \zeta)\overline{P_m^{(k)}(x,\zeta)} + \tau^{-1} I_m \sum P_{m,k}(x, \zeta)\overline{P_m^{(k)}(x, \zeta)} > 0$$

for all $\zeta = \xi + i\tau$ grad $\psi(x)$, with $\xi \in R_n$ and $0 \neq \tau \in R_1$, satisfying the equations

(2.6) $$P_m(x, \zeta) = 0, \quad \sum P_m^{(j)}(x, \zeta) \frac{\partial \psi}{\partial x_j} = 0.$$

Theorem 1. (Hörmander) Let Ω be a bounded open set, $\psi \in C^\infty(\bar{\Omega})$ be a real valued function and have strongly pseudo-convex level surfaces with respect to a principally normal operator P of order m and whose coefficients are in $L_\infty(\bar{\Omega})$ and those of P_m are in $C^2(\bar{\Omega})$. Then there is a constant K such that for $\varphi = e^{\lambda\psi}$ with sufficiently large λ and for sufficiently large τ

(2.7) $$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leqq \\ \leqq K \int |P(x, D)u|^2 e^{2\tau\varphi} dx, \quad u \in C_0^\infty(\Omega).$$

Theorem 2 (Hörmander). Let Ω and ψ fulfill the hypotheses of Theorem 1, and P be a elliptic operator of order m with coefficients in $L_\infty(\bar{\Omega})$ and those of P_m in $C^1(\bar{\Omega})$. Then there is a constant K such that for $\varphi = e^{\lambda\varphi}$ with sufficiently large λ and sufficiently large τ

(2.8) $$\sum_{|\alpha|\leqq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leqq \\ \leqq K \int |P(x, D)u|^2 e^{2\tau\varphi} dx, \quad u \in C_0^\infty(\Omega).$$

3. The estimates for operators with double characteristics. In this section we prove the following

Theorem 3. Let Ω be a bounded convex domain which contains the origin, $\psi \in C^{\infty}(\bar{\Omega})$ a real valued function with strongly pseudo-convex level surfaces with respect to P^{ν}, ($\nu=$ lor 2) where P^{ν} is an operator of order m_{ν} and fulfills the hypotheses of Theorem ν respectivly, and coefficients of P^2 are in $C^{m_1}(\bar{\Omega})$. Then there are constants K and ε such that for $\varphi=e^{\lambda\psi}$ with sufficiently large λ and sufficiently large τ

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|-1)} \int |D^{\alpha}u|^2 e^{2\tau\varphi}dx \leqq \tag{3.1}$$
$$\leqq \varepsilon^2 K \int |P(x, D)u|^2 e^{2\tau\varphi}dx, \quad u \in C_0^{\infty}(\Omega_{\varepsilon})$$

where $\Omega_{\varepsilon}=\{x; x/\varepsilon \in \Omega\}$, and $P(x, D)$ is the operator defined by (1.2).

First, we prove the following

Lemma. Let Ω, ψ, and P^1 fulfill the hypotheses of the Theorem. Then there are constants K and ε such that for $\varphi=e^{\lambda\psi}$ with sufficiently large λ and sufficiently large τ

$$\sum_{|\alpha|<m_1} \tau^{2(m_1-|\alpha|)-1} \int |D^{\alpha}u|^2 e^{2\tau\varphi}dx \leqq \tag{3.2}$$
$$\leqq \varepsilon^2 K \int |P^1(x, D)u|^2 e^{2\tau\varphi}dx, \quad u \in C_0^{\infty}(\Omega_{\varepsilon}).$$

Proof. Let $0<\varepsilon<1$ and set

$$P^1_{(\varepsilon)}(x, D)=P^1(\varepsilon x, D), \quad \psi_{\varepsilon}(x)=\psi(\varepsilon x), \ \varphi_{\varepsilon}(x)=\varphi(\varepsilon x). \tag{3.3}$$

Then we can easily show that $P^1_{(\varepsilon)}$ is principally normal and ψ_{ε} has strongly pseudo-convex level surfaces. Using Theorem 1, we get

$$\sum_{|\alpha|<m_1} \tau^{2(m_1-|\alpha|)-1} \int |D^{\alpha}u|^2 e^{2\tau\varphi_{\varepsilon}}dx \leqq \tag{3.4}$$
$$\leqq K \int |P^1_{(\varepsilon)}(x, D)u|^2 e^{2\tau\varphi_{\varepsilon}}dx, \quad u \in C_0^{\infty}(\Omega).$$

The inequality (3.2) follows easily from (3.4) by a stretching of the independent variables.

Remark. We wish to emphasize that K in (3.4) can be taken to be independent of ε. The proof of this fact follows easily from the proof of Theorem 1. But it is too long to describe it here.

(Proof of Theorem 3.) In view of (3.2) we have

$$\sum_{|\alpha|<m_1} \tau^{2(m_1-|\alpha|)-1} \int |D^{\alpha}P^2(x, D)u|^2 e^{2\tau\varphi}dx \leqq$$
$$\leqq \varepsilon^2 K \int |S(x, D)u|^2 e^{2\tau\varphi}dx, \quad u \in C_0^{\infty}(\Omega_{\varepsilon})$$

where $S(x, D)=P^1(x, D)P^2(x, D)$. Now $D^{\alpha}P^2(x, D)u=P^2(x, D)D^{\alpha}u+$ a linear combination with bounded coefficients of $D^{\beta}u$ with

$$|\beta| \leqq |\alpha|+m_2-1.$$

Since

$$m_1-|\alpha| \leqq m_1+m_2-|\beta|-1=m-|\beta|-1,$$

we get with another constant K

$$\sum_{|\alpha|<m_1} \tau^{2(m_1-|\alpha|)-1} \int |P^2(x, D)D^\alpha u|^2 e^{2\tau\varphi}dx \leqq$$
$$\leqq \varepsilon^2 K \int \Big\{ |S(x, D)u|^2 + \sum_{|\beta|<m-1} \tau^{2(m-|\beta|-1)-1} |D^\beta u|^2 \Big\} e^{2\tau\varphi}dx.$$

Using Theorem 2 in the left hand side and moving terms from right to left, we get with still another constant K

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|-1)} \int |D^\alpha u|^2 e^{2\tau\varphi}dx \leqq$$
$$\leqq \varepsilon^2 K \int |S(x, D)u|^2 e^{2\tau\varphi}dx.$$

$P(x, D)$ and $S(x, D)$ differ by terms of order $<m$. Thus we get

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|-1)} \int |D^\alpha u|^2 e^{2\tau\varphi}dx \leqq$$
$$\leqq \varepsilon^2 K \int \Big\{ |P(x, D)u|^2 + \sum_{|\alpha|<m} |D^\alpha u|^2 \Big\} e^{2\tau\varphi}dx.$$

We can move the term involving $D^\alpha u$ from right to left if $|\alpha|<m-1$, and as for the terms involving $D^\alpha u$ with $|\alpha|=m-1$, it is also possible to move them from right to left if we choose ε so small that $\varepsilon^2 K<1/2$. The proof is complete.

Remark. The proof shows that lower order terms are irrelevant for the validity of (3.1).

Example. Let

$$P^1(\xi)=\xi_1^2+\sum_2^n \xi_j^2, \quad P^2(\xi)=\xi_1^2-\sum_2^n \xi_j^2$$

and

$$\psi(x)=\sum_2^n x_j^2.$$

Then the level surface defined by

$$\psi(x)=\psi(x^0), \quad x^0=(0, 1, 0, \cdots, 0)$$

is strongly pseudo-convex with respect to P^ν, $(\nu=1$ or $2)$, at x^0, but not with respect to $P=P^1P^2$. See also Kumano-go [5].

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